Algebraic Approach to Competitive Equilibria*

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November 5, 2024

Abstract

We demonstrate that an efficient allocation of goods can be found in polynomial time if and only if a competitive equilibrium exists in the corresponding market. In other words, competitive markets solve precisely the class of all tractable optimization problems. This result holds under mild restrictions on allowable valuations and applies to markets involving both private and public goods. Our findings build on a novel connection between competitive equilibria and valued constraint satisfaction problems from complexity theory. We illustrate how this approach can be employed to establish the existence or nonexistence of competitive prices across various classes of valuations.

1 Introduction

When does a competitive equilibrium provably exist? The most well-known sufficient condition is the gross substitutes property (Kelso Jr and Crawford, 1982; Gul and Stacchetti, 1999), but it often proves cumbersome for practical applications (Brandenburg et al., 2022). We propose an alternative approach to identify classes of valuations for which competitive equilibria are guaranteed to exist by leveraging a novel connection with complexity theory.

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Bikhchandani and Ostroy (2002) demonstrated that in a package assignment problem, a competitive equilibrium exists if and only if a specific integer program can be solved via a linear programming relaxation. Remarkably, the linear program also renders an otherwise intractable integer program tractable due to the unimodularity of the constraint matrix. Unimodularity ensures that the integer constraints are redundant, allowing the solution to be found in polynomial time. This raises the intriguing question of whether the relationship between polynomial-time solvability and equilibrium existence extends to more general settings.

To this end, the present paper makes two contributions. On the theoretical side, we establish that markets can exactly solve the class of tractable optimization problems — those for which a polynomial-time algorithm exists. The direction from the existence of equilibrium to tractability is intuitive, as computing a competitive equilibrium is also a way of producing an efficient allocation. The converse fact that a competitive equilibrium is guaranteed to exist for any tractable problem is less obvious.

On the practical side, we provide a new procedure to identify classes of valuations and problems for which a competitive equilibrium is ensured. When designing an auction or market institution, a market designer must decide on a strategy for eliciting valuations or bids from buyers. She may opt to price individual items separately or employ a more complex bidding language (Nisan, 2000). She would then prefer a procedure that ensures that there are prices supporting an efficient auction outcome in equilibrium. Such a procedure would ensure that correct preferences are elicited and efficiency is maximized, assuming equilibrium behavior.

To illustrate our approach, we demonstrate that when buyers are allowed to bid on any single good with a corresponding willingness to pay, checking the existence of equilibria becomes particularly straightforward. To put this in the auction setting, we characterize equilibrium existence under the assumption that all singleton bids are permitted in the auction. Under this condition, the classes of valuations that always guarantee the existence of equilibria are precisely the supermodular valuations. For any other class of valuations, there exists a set of buyer preferences for which a competitive equilibrium fails to exist.

While the linear programming approach of Bikhchandani and Ostroy (2002) can verify the existence of equilibria in specific auction instances, this may not always be desirable. This approach implicitly assumes that the designer has already received the bids before checking for competitive prices. Instead, it is often preferable to restrict bids ex-ante to classes that are guaranteed to lead to competitive prices (see also Roughgarden and Talgam-Cohen (2015)). The most closely related work by Baldwin and Klemperer (2019) identifies unimodularity as one such restriction. We approach the problem from a complexity perspective, proposing a different criterion for the existence of equilibria.

Additionally, there are several well-known tractable cases where competitive equilibria are known to exist, such as: non-linear and non-anonymous pricing (Bikhchandani and Ostroy, 2002), graphical pricing (Brandenburg et al., 2022), gross substitutes (Gul and Stacchetti, 1999), unit bids (Lehmann et al., 2001), geometric structures (Rothkopf et al., 1998; Candogan et al., 2015), and tractable bidding languages (Nisan, 2000; Nisan and Segal, 2006). These are usually only sufficient but not necessary.

Our general model is also conceptually similar to a version of the Arrow-Debreu model of general equilibrium, known as the generalized Nash equilibrium problem (GNEP) (Facchinei and Kanzow, 2010), but with key differences: the discrete nature of choices and the inclusion of a numeraire good.

We begin by defining the instances and classes of markets within an abstract framework for allocating goods to maximize welfare, followed by the necessary algebraic definitions. The third and fourth sections present our main theorem and demonstrate its connection to complexity theory through illustrative examples. We then refine these results for the special case of conservative markets. We conclude by discussing avenues to relax the main algebraic assumption. All proofs are provided in the appendix.

2 Definitions

2.1 Economic concepts: markets

Let $Y = \{y_1, y_2, ...\}$ be a finite set of tradable goods. Let B denote a finite set of buyers. We define an allocation ω as a function mapping goods Y to an arbitrary set D, called the domain. The set of all such functions is denoted Ω. We are concerned with a problem of finding an allocation that maximizes the total utility of the buyers. The abstraction of the domain D is meant to capture both private and public goods. If $D = B$, then any allocation $\omega \in \Omega$ is a distribution of goods among the buyers. On the other hand, if $D = \{0, 1\}$, an allocation $\omega \in \Omega$ is a choice of a set of public goods to provide, i.e. the goods mapping to 1 are chosen.

Denote the set of all functions $f : D^n \to \mathbb{Q}$ for some finite $n \geq 1$ by $\mathcal{F}_n$ and let $\mathcal{F} = \bigcup_n \mathcal{F}_n$. For any function $f : D^n \to \mathbb{Q}$, n is said to be the arity of f. When we will have indexed functions, we will write n_t to denote the arity of function f_t. Each buyer $b \in B$ is assumed to have a rational-valued utility function over allocations $U_b : \Omega \to \mathbb{Q}$.

We first consider the problem of finding an allocation that maximizes the total utility of buyers $\arg\max_{\omega\in\Omega} \sum_{b\in B} U_b(\omega)$. We do not yet consider prices or equilibrium concepts. An instance of such problem is simply called an optimization problem.

Definition 1 (Optimization problem)**.** *An (instance of an) optimization problem is a collection $\langle B, Y, D, \{U_b\}_{b\in B}\rangle$, where B is a finite set of buyers, Y is a finite set of goods, $U_b : \Omega \to \mathbb{Q}$ is a utility function of buyer b, and Ω is the set of mappings $Y \to D$.*

Some optimization problems can be solved by establishing a market. Our concept of a market equilibrium is the following. We assume that the market designer can combine goods to be sold together as packages. Packages are used to elicit preferences of buyers, who decide on the set of packages to buy. The prices are linear – the price of a package is a sum of prices of goods that enter it. These prices determine how the revenue from selling a package would be split between sellers of individual goods. We formalize this below.

A package is an ordered subset of goods, represented by a mapping $x : \{1, 2, ..., n_x\} \to Y$. The integer n_x is called the size of the package and it is also the arity of x. We will denote the set of packages available in the market by X. Buyers pay for assigning goods in the package to values in the domain. We call these units of trade "lots". A lot specifies a part of an allocation: a mapping $d^x : \{1, 2, ..., n_x\} \to D$ from a package to the domain. We write D^{n_x} as the set of all such functions. [1]Formally, a lot is a tuple (x, d^x), where $x : \{1, 2, ..., n_x\} \to Y$ is a package and $d^x : \{1, 2, ..., n_x\} \to D$ is a function allocating the goods in the package to the domain.

Each buyer's utility is assumed to be additively separable in packages, and so buyers take buying decisions for every package independently and there is no exposure problem (see e.g. Meng and Gunay, 2017). Let $\omega^x = (\omega(x(1)), \omega(x(2)), ..., \omega(x(n_x)))$ for $\omega \in \Omega$ and $x \in X$. This vector is the restriction of allocation ω to goods in x. Then we assume that there are $\{u_b^x \in \mathcal{F}_{n_x}\}_{x \in X}$, such that $U_b(\omega) = \sum_{x \in X} u_b^x(\omega^x)$. We will sometimes abuse the notation and write $\omega^x(k)$ for k-th element of ω^x.

A subutility function specifies a utility level for every lot of a given package. This can be written more succintly as a bid. A bid (x, f) is a tuple consisting of a package x and a subutility function $f : D^{n_x} \to \mathbb{Q}$. A bid specifies the part of the utility for different assignments of goods in the package x.

We call a mapping $p : Y \times D \times X \times B \to \mathbb{Q}$ a price system. Each value $p_{x(i), d^x(i), x, b} \in \mathbb{Q}$ is the transfer from the buyer of package x to the seller of good $x(i)$ for assigning it the value $d^x(i)$, e.g. providing a public good or selling the good to a particular buyer. We do not restrict these transfers to be positive. Every lot (x, d^x) has an associated price, that a buyer b would have to pay for this part of the allocation to realize. It may be different for different buyers. This price equals $\sum_{1 \leq i \leq n_x} p_{x(i), d^x(i), x, b}$ Notice that the price is the sum of the prices of individual goods that enter the package, but may be different for different mappings d^x. In equilibrium, each buyer selects a set of lots to buy at these prices to maximize her

[1]This differs from the common definition of a package as simply a subset of goods. We consider a slightly more general model where the order in which goods enter the package also matters.

utility. Buyers are not constrained by their budgets and have quasilinear utility. The buyer's problem is to find

$$\arg\max_{S \subseteq \{(x,d^x): x \in X, d^x \in D^{n_x}\}} \sum_{(x,d^x) \in S} \left(u_b^x(d^x) - \sum_{1 \le i \le n_x} p_{x(i),d^x(i),x,b} \right).$$

Every good has a seller, who is similarly maximizing utility. The utility of the sellers comes from selling packages containing their goods. The problem of the seller of good $y \in Y$ is then to find

$$\arg\max_{d} \sum_{x \in X : y \in Im(x)} p(y,d,x,b).$$

This value is the sum of all transfers from buyers of packages that contain good y. If $D = B$, then the maximization becomes the problem of choosing the buyer that would pay the maximum total price over all packages containing y. If $D = \{0,1\}$, then it becomes instead the decision to provide the public good.

A market is an optimization problem, where for every buyer the utility function over allocations is replaced by utility over packages.

Definition 2 (Market)**.** *A market is a collection* $\langle B, Y, D, X, \{(x, u_b^x)\}_{(b,x) \in B \times X} \rangle$*, where* B *is a finite set of buyers,* Y *is a finite set of goods,* $X \subseteq 2^Y$ *is a set of packages,* $u_b^x \in \mathcal{F}_{n_x}$ *is a subutility function of buyer* i *for package* x*.*

Every market $\langle B, Y, D, X, \{(x, u_b^x)\}_{(b,x) \in B \times X} \rangle$ has an associated optimization problem $\langle B, Y, D, \{U_b\}_{b \in B} \rangle$ of finding an optimal allocation $\arg\max_{\omega \in \Omega} \sum_{b \in B} U_b(\omega)$, where $U_b(\omega) = \sum_{x \in X} u_b^x(\omega^x)$.

2.2 Classes of markets

The designer will usually not know the exact utility functions of the buyers. Instead, the designer would know that they belong to a certain class, and her goal is to ensure that the competitive equilibrium exists for all possible markets in the class. A market class

$Market(\mathcal{U})$ is defined by a set of functions $\mathcal{U} \subseteq \mathcal{F}_D$. This set does not depend on the number of goods and functions in it can be of any (finite) arity. A class describes possible relationships between goods, but does not restrict their overall number.

We say that a market $\langle B, Y, D, X, \{(x, u_b^x)\}_{(b,x)\in B\times X}\rangle$ is in the market class $Market(\mathcal{U})$ if any instance in the class uses only the functions from $\mathcal{U}$: $u_b^x \in \mathcal{U}$ for all $b \in B$, $x \in X$.

One of the consequences of this algebraic setup is that there is no information about packages stored in the market class. It only defines subutility functions of some finite arity, but does not declare which packages they will be applied to. For example, the following function $\sigma^1 \in \mathcal{F}_2$ encodes a complementarity relationship between two goods x_1 and x_2:

$$\sigma^1(x_1, x_2) = \begin{cases} 1 & \text{if } x_1 = x_2 \\ 0 & \text{otherwise} \end{cases}$$

The value of the function is positive only if the goods are assigned the same value, e.g. sold to the same buyer $b \in D = B$. However, x_1 and x_2 could correspond to any goods in Y. An instance of $Market(\sigma^1)$ might contain a bid $((y_1, y_2), \sigma)$ or a bid $((y_2, y_3), \sigma)$, or neither, or both. That is, there is no intrinsic difference between goods on the class level and all the functions are neutral in terms of goods. The presence of function σ only says that buyers may treat a pair or pairs of goods as complements, but it does not say that they do and which. For clarity of discussion, we make this assumption explicit.

Assumption 1 (Algebraicity). *If a market $\langle B, Y, D, X, \{(x, u_b^x)\}_{(b,x)\in B\times X}\rangle$ is in the class $Market(\mathcal{U})$, then also $\langle B, Y, D, X', \{(x', u_b^x)\}_{(b,x')\in B\times X'}\rangle$ is in the same class $Market(\mathcal{U})$ for any $X' \subseteq 2^Y$, $|X'| = |X|$.*

Algebraicity roughly says that if buyers value certain packages of goods in a certain way, other buyers in a different scenario might value different goods in the same way. If we interpret the market as a combinatorial auction then algebraicity says that if a certain bid on a certain combination of goods is allowed, then the same bid is allowed on other combinations of the same size.

The designer is interested in knowing whether the optimal allocation can be supported by a competitive equilibrium. We define competitive equilibrium as a pair of an optimal allocation $\omega \in \Omega$ and a price system $p : Y \times D \times X \times B \to \mathbb{Q}$, such that all players solve their respective optimization problems.

Definition 3. *A* competitive equilibrium *for a market is a pair* (ω, p) *of an allocation* $\omega \in \Omega$ *and a price system* $p : Y \times D \times X \times B \to \mathbb{Q}$*, such that*

1. *buyers optimize: for all* $b \in B$ *and* $S \subseteq \{(x, d^x) : x \in X, d^x \in D^{n_x}\}$*,*

$$U_b(\omega) = \sum_{x \in X} \left(u_b^x(\omega^x) - \sum_{1 \leq i \leq n_x} p_{x(i),\omega^x(i),x,b} \right) \geq \sum_{(x,d^x) \in S} \left(u_b^x(d^x) - \sum_{1 \leq i \leq n_x} p_{x(i),d^x(i),x,b} \right).$$

2. *sellers optimize: for all* $y \in Y$ *and* $d \in D$*,*

$$\sum_{x \in X : y \in Im(x)} p(y, \omega(y), x, b) \geq \sum_{x \in X : y \in Im(x)} p(y, d, x, b).$$

3. *For any* $y \in Y$*,* $x \in X$*, and* $b \in B$ *there is* $d \in D$*, s.t.* $p(y, d, x, b) = 0$*.*

The first condition says that no buyer wants to buy a different set of lots than the one induced by the equilibrium allocation ω. The second condition ensures that no seller wants to assign her good to a different value in D for a higher revenue. The last condition says that there is a choice for every good that does not require any buyer to pay anything. This condition is the standard requirement of individual rationality: a choice where no money changes hands is always available to every buyer and every seller. Since the packages are priced linearly, this condition implies that there is also always a bid (x, d^x) for every package $x \in X$ that has a zero price. Formally, $\sum_{1 \leq i \leq n_x} p_{x(i),d^x(i),x,b} = 0$, where $d^x(i) \in D$ is such that $p_{x(i),d^x(i),x,b} = 0$ for every $1 \leq i \leq n_x$.

2.3 Complexity and algebraic concepts

The main results rely on the concept of a valued constraint satisfaction problem (VCSP) and its properties. Let Z denote the set of assignments, maps $V \to D$. This is the same construction as the set of allocations for the markets. Then we can define a finite-valued VCSP as follows.

Definition 4. *An instance of a binary finite-valued constraint satisfaction problem is a triple $\langle V, D, f \rangle$ where:*

- *V is a finite set of variables,*
- *D is a finite set of values,*
- *f is the objective function, mapping assignments of variables Z to $\mathbb{Q}$. It is given by*

$$f(z) = \sum_{t \in T} f_t \left(z_{i_t(1)}, \ldots, z_{i_t(n_t)} \right),$$

with T standing for a finite set. Each constraint tuple (i_t, f_t) is specified by a function f_t of arity n_t and indices $i_t(k), k = 1, \ldots, n_t$.

Such VCSP are called finite-valued, because the values of all constraint functions f_t are finite. In general-valued VCSP there may also be hard constraints that take value $-\infty$. For our market problems that would imply that some allocations are infeasible. In this paper we assume that all allocations in ω are feasible and therefore only consider finite-valued VCSP.

A solution to an instance of VCSP is an assignment $z : V \to D$ with the maximum value of the objective function f[2]. VCSP classes are defined in terms of the allowed relationships between variables and the kinds of constraints that can be present. A *valued constraint language* $\Gamma \subseteq \mathcal{F}$ is a set of cost functions, and $VCSP(\Gamma)$ is the class of all VCSP instances in which all cost functions are from Γ.

[2]We use a more suggestive formulation as a maximization problem rather than minimization, which is the standard in the constraint satisfaction literature

The presence of indices in the definition of the objective function f crucially restricts cost functions. Similarly to market classes, cost functions hold no information about different input variables, only *instances* of VCSP have that information. For example, $\Gamma = \{f_1(\cdot,\cdot)\}$ where f_1 is some function of arity $n_1 = 2$, $V = \{z_1, z_2, z_3, z_4\}$ implies that any of following versions of f_1 can be present in the instance: $f_1(z_1, z_2)$, $f_1(z_3, z_4)$, $f_1(z_2, z_4)$.[3]

We will say that a valued constraint language Γ is said to be tractable if every instance in $VCSP(\Gamma)$ can be solved in polynomial time. Otherwise it is said to be intractable. A specific algorithm for solving VCSP that will be important in our proofs is optimization. The following linear program is called the basic LP relaxation (BLP) of a VCSP instance:

$$\begin{array}{lll} \max & \sum_{t\in T}\sum_{z^t\in D^{n_t}} f_t\left(z^t\right)\lambda_{t,z^t} & \\ \text{s.t.} & \sum_{\substack{z^t\in D^{n_t}\\ z^t(k)=d}} \lambda_{t,z^t} = \mu_{i_t(k),d}, & \forall t\in T, k\in\{1,...,n_t\}, d\in D \\ & \sum_{d\in D}\mu_{v,d} = 1 & \forall v\in V \\ & \sum_{z^t\in D^{n_t}}\lambda_{t,z^t} = 1 & \forall t\in T \\ & 0\le \lambda,\mu\le 1 & \end{array} \tag{1}$$

The variables in the program are $\mu_{v,d}$ for every $v \in V$, $d \in D$ and λ_{t,z^t} for every $t \in T$ and $z^t \in D^{n_t}$. If all of these variables were constrained to be integer, the solution would correspond exactly to the solution of the VCSP instance. Without integrality constraints, the solution may be non-integer, and then this program only offers an approximation. The variables $\mu_{v,d}$ and λ_{t,z^t} can be interpreted as probability distributions on D and D^{n_t} respectively. The first constraint relates the marginal probability of λ to the distribution of μ, and the rest of the constraints ensure feasibility and that the distributions are well-defined.

[3]Such constraints are called relational, and we only consider these types of constraints. The counterpart, structural constraints that restrict the scopes and compositions of functions are an active field of study. Combinations of the two types of constraints are called hybrid constraints Cooper and Živnỳ (2011).

3 General result

Our main result is the if-and-only-if connection between the tractability of markets and existence of competitive equilibria.

Proposition 1. *Every market in the class $Market(\mathcal{U})$ has a competitive equilibrium if and only if every corresponding optimization problem can be solved in time polynomial in the number of goods Y.*

The first step of the proof relies on the following seminal dichotomy result by Thapper and Živnỳ (2016) for finite-valued constraint languages.

Theorem 1 (Thapper and Živnỳ 2016)**.** *Let D be an arbitrary finite set and let Γ be a finite-valued constraint language defined on D. $VCSP(\Gamma)$ is tractable if and only if the BLP solves $VCSP(\Gamma)$. Otherwise, $VCSP(\Gamma)$ is NP-hard.*

The second step uses the fact that the BLP itself has a direct connection to the existence of competitive equilibria, which can be characterized through the dual of this linear program and complementary slackness. This is a similar argument to the one found in Bikhchandani et al. (2001) and Bikhchandani and Ostroy (2002) for package auctions. The complete proof can be found in the Appendix.

Proposition 1 suggests a way to test a specific market for competitive equilibrium. If it can be formulated using the functions from some tractable class, then the equilibrium exists. This can often be done efficiently through existing dichotomy results, which we will illustrate in section 5 for conservative languages.

Corollary 1. *A market $\langle B, YF, \{U_b\}_{b\in B}\rangle$ has a competitive equilibrium if it belongs to some class $Market(\mathcal{F},\mathcal{U})$, where for every market the optimal allocation can be found in polynomial time.*

The converse is not true. A tractable problem typically belongs to both tractable and non-tractable classes.

4 Examples

In all examples, graph illustrations of markets will be useful. The vertices of these graphs are the goods, and the edges are packages of two goods. We denote the graph G. The set of vertices is the set of goods Y. The set of edges of this graph is denoted $E(G) \subseteq Y^2$. The weights of the edges encode the extra utility that can be gained from the package consisting of the goods that are connected by the edge.

4.1 Cutlery auction

First, consider an example with complementary goods called the cutlery auction (Brandenburg et al., 2022). There are 3 goods $Y = \{S, K, F\}$ and 3 buyers $B = \{Steak, Fruit, Spaghetti\}$. The objects are a fork (F), a knife (K), and a spoon (S). The three buyers have different favorite combinations and derive a utility of 1 only if they receive their favourite combination. Otherwise their utility is zero. The Steak buyer only gets positive utility from a combination of a fork and a knife, Spaghetti buyer desires a fork and a spoon, and Fruit – a knife and a spoon. Each of these combinations is a package in our framework. The cutlery auction is illustrated by the graph G in Figure 1, the three goods are the labeled vertices of the graph, and the edges encode the packages. There is a gain in total utility of 1 if both goods at the vertices are assigned to the buyer with the corresponding favorite combination.

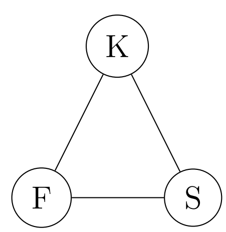

Figure 1: Cutlery auction

This type of preferences can be captured in our framework by means of a function that encodes complementarity. First, let the domain D be the set of buyers B with the interpretation that if $\omega(y) = b, b \in B$, then buyer b obtains the good. For each pair of complement goods, we would use subutility functions $\phi^{C,i} \in \mathcal{F}_2$, defined as follows:

$$\phi^{C,i}(x_1, x_2) = \begin{cases} C & \text{if } x_1 = x_2 = i \\ 0 & \text{otherwise} \end{cases}$$

for some $C \in \mathbb{Q}_+$ and $i \in D$. The collection of such functions for all possible values of $C \in \mathbb{Q}_+$ and $i \in D = B$ is denoted $\phi = \{\phi_{C,i}\}_{(C,i) \in \mathbb{Q}_+ \times B}$.

The class $Market(\phi)$ is the class of all markets where players derive value from obtaining pairs of goods. Each such market is given by a graph G and bids $((x_1, x_2), \phi^{C,i})$ for every edge $(x_1, x_2) \in E(G)$, the buyer i and utility gain C. The markets in this class do not always have competitive equilibria. In particular, the specific instance in Figure 1 does not have one. It would consist of bids $\big((F, K), \phi^{1,Steak}\big)$, $\big((S, K), \phi^{1,Fruit}\big)$, and $\big((S, F), \phi^{1,Spaghetti}\big)$.

In any equilibrium, the allocation would have to be efficient. In this market this means that the overall utility is 1 and exactly one favorite combination is provided to some buyer. However, the BLP for this instance without integrality constraints can yield a utility of 1.5 by setting $\mu_{K,Steak} = \mu_{K,Fruit} = 0.5$, $\mu_{S,Spaghetti} = \mu_{S,Fruit} = 0.5$, $\mu_{F,Steak} = \mu_{F,Spaghetti} = 0.5$, $\mu_{K,Spaghetti} = \mu_{F,Fruit} = \mu_{S,Steak} = 0$. Thus, BLP does not solve the cutlery auction. By a combination of Theorem 1 and Proposition 1, this market class is intractable and competitive equilibria will sometimes not exist. This implication can be seen directly by inspecting the following linear program that is dual to the BLP:

$$
\begin{aligned}
\min \sum_{y \in Y} \pi_y^S + \sum_{b \in B} \pi_b^{B,\phi}, & \\
\text{s.t. } \pi_K^S \geq p_{K,d,Steak} + p_{K,d,Fruit} & \quad \forall d \in B \\
\pi_S^S \geq p_{S,d,Spaghetti} + p_{S,d,Fruit} & \quad \forall d \in B \\
\pi_F^S \geq p_{F,d,Steak} + p_{F,d,Spaghetti} & \quad \forall d \in B \\
\pi_{Steak}^{B,\phi} + p_{K,Steak,Steak} + p_{F,Steak,Steak} \geq 1 & \\
\pi_{Steak}^{B,\phi} + p_{K,d_1,Steak} + p_{F,d_2,Steak} \geq 0 & \quad \forall (d_1, d_2) \in B^2 \setminus Steak^2 \\
\pi_{Fruit}^{B,\phi} + p_{K,Fruit,Fruit} + p_{S,Fruit,Fruit} \geq 1 & \\
\pi_{Fruit}^{B,\phi} + p_{K,d_1,Fruit} + p_{S,d_2,Fruit} \geq 0 & \quad \forall (d_1, d_2) \in B^2 \setminus Fruit^2 \\
\pi_{Spaghetti}^{B,\phi} + p_{S,Spaghetti,Spaghetti} + p_{F,Spaghetti,Spaghetti} \geq 1 & \\
\pi_{Spaghetti}^{B,\phi} + p_{S,d_1,Spaghetti} + p_{F,d_2,Spaghetti} \geq 0 & \quad \forall (d_1, d_2) \in B^2 \setminus Spaghetti^2,
\end{aligned}
$$

where $Steak^2$ is shorthand for $(Steak, Steak)$, etc.

The complementary slackness conditions are:

$$
\begin{aligned}
\mu_{y,d} > 0 \implies \pi_y^S = \sum_{(b,x) \in B \times X} p_{y,d,x,b} & \quad \forall y \in Y, d \in B \\
\lambda_{t,d^x} > 0 \implies \pi_t^B = u_b^x(d^x) - \sum_{y \in Y} p_{y,d^x(y),x,b} & \quad \forall t = (b,x) \in B \times X, d^x \in D^{n_x}
\end{aligned}
$$

The dual program together with the complementary slackness encodes the competitive equilibrium conditions, except for the normalization (condition three in Definition 3). The p variables correspond to the price system with omitted fourth index since every buyer has only one corresponding bid, i.e. is interested in only one package. The variables π^B correspond to subutilities of buyers from buying different packages, and π^S to utilities of sellers. In any efficient allocation at least two goods are assigned to the same buyer. The total utility is 1, which can be shared between buyers and sellers through the choice of prices.

Without loss of generality, let F and K be assigned to $Steak$. By complementary slackness, $0 \leq p_{K,Steak,Steak} + p_{F,Steak,Steak} \leq 1$.

By the third condition in Definition 3 there is a d, s.t. $p_{y,d,b} = 0$. Then by the dual program, $\pi_b^{B,\phi} \geq 0$ and $\pi_y^S \geq 0$. This, in turn, implies that $p_{S,d,Spaghetti} = p_{S,d,Fruit} = 0$ for all $d \in B$. Combining this with the fact that $0 \leq p_{K,Steak,Steak} + p_{F,Steak,Steak} \leq 1$, we know that either $p_{K,Fruit,Fruit} + p_{S,Fruit,Fruit} < 1$ or $p_{S,Spaghetti,Spaghetti} + p_{F,Spaghetti,Spaghetti} < 1$. Therefore $\pi_{Fruit}^{B,\phi} > 0$ or $\pi_{Spaghetti}^{B,\phi} > 0$. Suppose $\pi_{Fruit}^{B,\phi} > 0$. Then by complementary slackness $\lambda_{(Fruit,(K,S)),(d_1,d_2)} = 0$ for any $(d_1, d_2) \in B^2 \setminus Fruit^2$. So $\lambda_{(Fruit,(K,S)),(Fruit,Fruit)} = 1$. Since at the same time $\lambda_{(Steak,(K,F)),(Steak,Steak)} = 1$, this is not possible for any integer values of μ.

Therefore there is no competitive equilibrium for the specific market instance of the cutlery auction. This connection between the dual and the competitive equilibrium is what gives rise to Proposition 1. It allows us to say that the competitive equilibrium is not guaranteed to exist without looking for the counterexample or analysing the linear programs as above. Indeed, $Market(\phi)$ includes the multiway cut problem, which is known to be NP-hard. This fact combined with Proposition 1 already says that competitive equilibrium will not exist for all markets in $Market(\phi)$.

4.2 Complements

The previous example had extreme differences in preferences among the three buyers: what some considered complements, others considered worthless. In this section we show that even if the three buyers would always agree on the complementarity of goods, competitive equilibrium is not guaranteed to exist. However, it always exists for such a market with only two buyers.

Consider a market where each buyer views certain pairs of tradeable goods from Y as complements. Their total utility is the sum of utilities of all goods that each buyer acquires and an extra utility for some pairs of goods if they are both acquired by the same person. We again use a weighted undirected graph G with a vertex for every good to model complementarities. We add an edge between every pair of goods with complementarity. The weight of that edge corresponds to the extra total utility gained by obtaining both goods at its ends by any player.

This example can be captured in our framework by means of the following functions. Once again, let the domain D be the set of buyers B. First, individual goods yield some utility to the buyer:

$$\psi^{C,i}(x_1) = \begin{cases} C & \text{if } x_1 = i \\ 0 & \text{otherwise} \end{cases}$$

for some $C \in \mathbb{Q}_+$ and $i \in D = B$. The collection of such functions for all possible values of $C \in \mathbb{Q}_+$ and $i \in D = B$ is denoted $\psi_{|B|} = \{\psi^{C,i}\}_{(C,i)\in\mathbb{Q}_+\times B}$. We make a distinction between $\psi_{|B|}$ for different $|B|$, because we will show that equilibrium existence depends on the number of buyers.

The buyer has to also get an extra utility of C if she obtains both goods. For each pair of complement goods, we impose a binary constraint on the corresponding two variables $\sigma^C \in \mathcal{F}_2$, defined as follows:

$$\sigma^C(x_1, x_2) = \begin{cases} C & \text{if } x_1 = x_2 \\ 0 & \text{otherwise} \end{cases}$$

for some $C \in \mathbb{Q}_+$. Similarly to $\psi_{|B|}$, the collection of such functions for all possible values of $C \in \mathbb{Q}_+$ is denoted $\sigma = \{\sigma_C\}_{C\in\mathbb{Q}_+}$.

The language for this example is then $\mathcal{U} = \sigma \cup \psi_{|B|}$. As in the previous example, any market where buyers have either additively separable utilities or derive extra utility from some pairs of goods (same price for everyone) can be modeled using the functions in $\mathcal{U}$. Any instance would contain a function from $\psi_{|B|}$ for additively-separable subutility of each good and a function from σ for any complementarities between pairs of goods. Maximizing the total utility of $|B|$ buyers with utilities from this class is equivalent to the following graph problem. There are two components of the objective function - the weights of edges between vertices of different colors and the values of different color assignments for each vertex. The goal is to color the vertices of this graph with $|B|$ colors, so as to maximize the sum of these two components.

The classical problem of finding a minimum cut in a graph is a special case of this problem. This class of problems is known in theoretical computer science as the Ising or, more generally, Potts model with external field (Gridchyn and Kolmogorov, 2013), originally used in statistical mechanics to explain ferromagnetism. This problem is known to be polynomially tractable for $|B| = 2$. Proposition 1 then implies that every market in $Market(\sigma \cup \psi_2)$ has a competitive equilibrium.

However, with $|B| > 2$, the problem is intractable as it can encode the NP-Hard multiway cut problem as a special case. Therefore, again by our Proposition 1, the competitive equilibrium for markets in $Market(\sigma \cup \psi_3)$ is no longer guaranteed to exist. Specifically, there is a market with buyers who have utilities from this class where the competitive equilibrium does not exist. The market illustrated in Figure 2 is such a market. All weights are either 1 (if not marked) or 4 (if marked). Further, the three vertices labelled y_1, y_2, y_3 are particularly desired by players $1, 2$ and 3 respectively for a utility of 1. The other goods only create value in combination with other goods. The overall problem consists of $(y_i, \sigma^{1,i})$ for $i \in \{1, 2, 3\}$ and $((x_1, x_2), \phi^1)$ or $((x_1, x_2), \phi^4)$ for every $(x_1, x_2) \in E(G)$.[4]

It is trivial, if tedious, to check that in this example the linear program and the integer program solutions also do not coincide.[5] However, there is also a structural reason for this difference between $|B| = 2$ and $|B| > 2$. The functions in the model with $|B| = 2$ are supermodular, while for $|B| > 2$ they are not. In Section 5 we show that when ψ is present among the subutility functions, supermodularity is the exact condition for tractability and, by means of Proposition 1, existence of equilibria.

[4]The example is based on Papadimitriou et al. (2008), specifically it is the modified Graph C with an additional edge.

[5]Python code for generating counterexamples and checking the values of the BLP and the integer programs is available in the supplementary material.

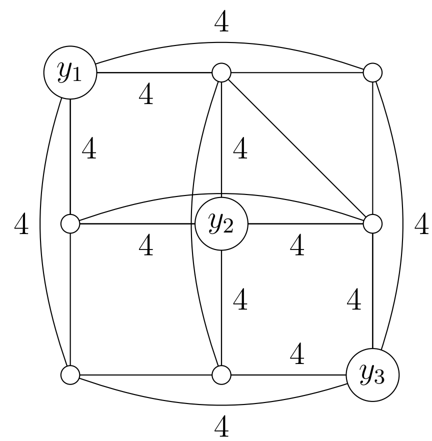

Figure 2: A market with complementary preferences and without a competitive equilibrium

4.3 Public goods

We use the last example to illustrate substitute valuations with public goods. Consider the problem of allocating a subset of public goods, when buyers derive positive utility from some public goods, but combinations of goods may be harmful. This captures a market with negative externalities between two goods, when some surplus is destroyed when both public goods are in operation.

For this example, the domain is binary $D = \{0, 1\}$. The interpretation of a good y being assigned a value 1, $\omega(y) = 1$, is that the good was allocated. If the good is assigned the value 0 the good was not allocated.

We again represent the relationships in the market using an undirected graph G. For each edge $(x_1, x_2) \in E(G)$, add the package with the goods corresponding to the two vertices of the edge and the subutility function $\mu \in \mathcal{F}_2$, defined as follows:

$$\eta^C(x_1, x_2) = \begin{cases} -C & \text{if } x_1 = x_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

for some $C \in \mathbb{Q}_+$. Let $\eta = \{\eta^C\}_{C \in \mathbb{Q}_+}$.

For simplicity, we further assume that the goods by themselves have value either zero or one to each buyer. If the value is one the following subutility function is added in combination with a singleton package containing the corresponding good.

$$\psi^{1,1}(x) = \begin{cases} 1 & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases}$$

Even these two types of functions alone are already intractable as $Market(\eta, \psi^{1,1})$ includes the NP-Hard problem of finding a maximum independent set, the largest set of vertices that are not connected to each other. Proposition 1 says that there is a market with such preferences for which competitive equilibrium does not exist.

5 Application: Conservative markets

For a natural class of languages, called conservative, a simple condition is both necessary and sufficient for existence of competitive equilibria. A language Γ is called *conservative* if it contains all unary functions $\mathcal{F}_1$. Similarly, we call a market class $Market(\mathcal{U})$ conservative if $\mathcal{U}$ contains all unary functions $\mathcal{F}_1$. Namely, the set of all unary functions for domain D is $\psi_{|D|}$ (the $\mathbb{Q}_+$ codomain is without loss of generality).

The markets corresponding to conservative VCSP allow buyers to have additively separable utility functions, so that a constant utility is derived from every good independently. Conservativeness does not restrict the buyers to have only this types of utility, but ensures that these utility functions can be represented (among others). In many if not most economic situations it is unreasonable to preclude the buyers from having simple additively-separable preferences.

We will show that for conservative markets, existence of competitive equilibria reduces to supermodularity. A function $f : D^k \to \mathbb{Q}$ is called (super-) submodular on the lattice $L = (D; \wedge, \vee)$ if $f(a \wedge b) + f(a \vee b) \leq (\geq) f(a) + f(b)$ for all $a, b \in D^k$. A set of functions Γ is said to be (super-) submodular on some lattice L if every function in Γ is (super-) submodular on L. If L is also a total order, then it is called a chain. Minimization of such submodular functions is known to be tractable (Iwata et al., 2001). The following Theorem 3.6 by Kolmogorov and Živný (2013) states that for conservative finite-valued languages, these are the only tractable functions:

Theorem 2 (Kolmogorov and Živný (2013))**.** *If a conservative finite-valued language is supermodular on some chain, then it is tractable. Otherwise it is NP-hard.*

Proposition 2. *If $\psi_{|D|} \subseteq \mathcal{U}$, i.e. $\mathcal{U}$ contains all finite-valued unary functions, then every market in the class $Market(\mathcal{U})$ has a competitive equilibrium if and only every function in $\mathcal{U}$ is supermodular on some chain.*

In other words, if there is a non-supermodular function in $\mathcal{U}$, then there is a market in $Market(\mathcal{U})$ that has no competitive equilibrium.

6 Discussion

This paper takes initial steps in exploring the three-way connection between complexity theory, linear programming, and competitive equilibria. Our aim is to inform market design by identifying classes of valuations that ensure the existence of equilibria. Proposition 1 is a step in this direction, offering an alternative view on the relationship between tractability and equilibria beyond unimodularity.

Proposition 2 can also be seen as a negative result. Additively separable utility functions can rarely be excluded, and supermodularity offers only a very narrow possibility for existence of competitive equilibria. This result is discouraging for the market designer and suggests that relaxing the algebraic condition might be a desirable extension of the analysis.

Moreover, several classes of valuations known to admit equilibria do not align with the algebraic structures considered here. For instance, assignment markets as studied by Shapley and Shubik (1971) rely on bipartite structures, which require structural constraints that are not present in our framework. Exploring this "other side" (Carbonnel et al., 2022) of valued constraint satisfaction problems could yield further positive results for new classes of valuations.

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Proof of Proposition 1. Take any market $\langle B, Y, D, X, \{(x, u_b^x)\}_{(b,x)\in B\times X}\rangle$ in the market class $Market(\mathcal{U})$. Construct a VCSP instance $\{V, D, f\}$ from the market problem, where $V = Y$ and $f(z) = \sum_{t\in T} f_t\left(z_{i(t,1)}, \ldots, z_{i(t,n_t)}\right)$, $T = B \times X$, and there is a constraint tuple (i_t, f_t) for each bid (x, u_b^x).

The BLP (eq. 1) for this VCSP can be written as:

$$\begin{aligned}
\max \quad & \textstyle\sum_{(b,x)\in B\times X} \sum_{d^x\in D^{n_x}} u_b^x\left(d^x\right) \lambda_{t,d^x}, & \\
\text{s.t.} \quad & \textstyle\sum_{\substack{d^x\in D^{n_x}\\ d^x(k)=d}} \lambda_{t,d^x} = \mu_{x(k),d}, & \forall t = (b,x) \in B\times X, k\in\{1,...,n_x\}, d\in D \\
& \textstyle\sum_{d\in D} \mu_{y,d} = 1 & \forall y \in Y \\
& \textstyle\sum_{d^x\in D^{n_x}} \lambda_{t,d^x} = 1 & \forall t = (b,x) \in B\times X, d^x \in D^{n_x} \\
& 0 \le \lambda, \mu \le 1 &
\end{aligned}$$

The dual linear program for this instance is:

$$\begin{aligned}
\min \quad & \textstyle\sum_{y\in Y} \pi_y^S + \sum_{t=(b,x)\in B\times X} \pi_t^B, & \\
\text{s.t.} \quad & \pi_y^S \ge \textstyle\sum_{=(b,x)\in B\times X} p_{y,d,x,b} & \forall y\in Y, d\in D \\
& \pi_t^B + \textstyle\sum_{y\in Y} p_{y,d^x(y),x,b} \ge u_b^x(d^x) & \forall t=(b,x)\in B\times X, d^x\in D^{n_x}
\end{aligned}$$

The complementary slackness conditions stemming from these linear programs are:

$$\mu_{y,d} > 0 \implies \pi_y^S = \sum_{(b,x)\in B\times X} p_{y,d,x,b} \qquad \forall y\in Y, d\in D \qquad \text{(CS-1)}$$

$$\lambda_{t,d^x} > 0 \implies \pi_t^B = u_b^x(d^x) - \sum_{y\in Y} p_{y,d^x(y),x,b} \qquad \forall t=(b,x)\in B\times X, d^x\in D^{n_x} \qquad \text{(CS-2)}$$

The non-trivial conditions from the combination of the primal, the dual, and the complementary slackness correspond exactly to the first two conditions of the competitive equilibrium (Definition 3). Specifically, suppose that the VCSP defined by the market can be solved in polynomial time. By Theorem 1, a solution to the BLP exists. Denote the optimal assignment of primal variables by $\mu_{y,d}$ and λ_{t,d^x} and the dual variables by $p_{y,d,t}$, π_y^S and π_t^B

for $d \in D, y \in Y, t \in T, d^x \in D^{n_x}$. Set $\omega(y) = d$ iff $\mu_{y,d} = 1$. Since BLP solves the VCSP instance, the solution to it is integral. This implies that the constructed ω is a well-defined allocation by the constraints of the primal problem. The first two conditions for equilibrium also hold: optimization by buyers is CS-2 combined with the second dual constraint, and optimization by sellers is CS-1 combined with the first dual constraint.

We now show that the last, third condition is without loss of generality. For any $y \in Y$, $x \in X$, and $b \in B$, set $p'_{y,d,x,b} = p_{y,d,x,b} - \min_d p_{y,d,x,b}$. Then $p'_{y,d',x,b} = 0$ for $d' = \arg\min_d p'_{y,d,x,b}$ as required. Let $\pi_y'^S = \pi_y^S - \sum_{(b,x)\in B\times X} \min_d p_{y,d,x,b}$ for every $y \in Y$ and $\pi_t'^B = \pi_t^B + \sum_{y\in Y} \min_d p_{y,d,x,b}$ for every $t = (b,x) \in B \times X$. If the vector of variables $(\{\pi_t^B\}_{t=(b,x)\in B\times X}, \{\pi_y^S\}_{y\in Y}, \{p_{y,d,x,b}\}_{y\in Y, d\in D, b\in B, x\in X})$ is the solution to the dual program, then so is $(\{\pi_t'^B\}_{t=(b,x)\in B\times X}, \{\pi_y'^S\}_{y\in Y}, \{p_{y,d,x,b}'^S\}_{y\in Y, d\in D, b\in B, x\in X})$. The objective function, the primal solution (and therefore ω) are unchanged and the constraints are still satisfied. This is the consequence of transferable utility in this model – change in prices only changes the distribution of utility between π_t^B and π_y^S. The new prices p' satisfy the third requirement in Definition 3, and so the pair $(\omega, \hat{p}')$ is the desired competitive equilibrium.

Suppose now that the VCSP cannot be solved in polynomial time, and in particular by this BLP. Then the constraints of the dual are not satisfied at the integer solution to the primal. Since these conditions correspond to the conditions for competitive equilibrium, there is no pair of an allocation and a price vector that would be a competitive equilibrium. □

Proof of Proposition 2. The result follows immediately by combining Proposition 1 with Theorem 2. □