Submitted to *International Economic Review*

ASSIGNMENT MARKETS: THEORY AND EXPERIMENTS

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We model a two-sided market for heterogeneous indivisible goods as a strategic game and as a bargaining game, and contrast core predictions with those of a Nash equilibrium and a Nash bargaining solution. We test the predictions on several institutions induced in the experimental laboratory. These simple static models prove to be able to explain market outcomes and predict behavior while circumventing the complexity of this dynamic environment. Moreover, the performance of the competing theories reflects the differences in institutions and negotiation procedures—we observe market outcomes close to Nash equilibrium predictions under auction-like institutions, and close to generalized bargaining for institutions that feature decentralized communication. This difference may be driving the documented effect of free-form bargaining reducing the number of no-trade outcomes at the expense of a higher chance of suboptimal match.

KEYWORDS: assignment, housing market, bargaining, laboratory experiment.

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The authors thank Jordi Brandts, Heinrich Nax, Ryan Tierney, and several audiences for useful comments.

1. INTRODUCTION

The assignment game represents markets characterized by indivisibility and heterogeneity of goods, and inflexibility of supply and demand satiated at exactly one unit. In a classic contribution, Shapley and Shubik (1971) proved that core allocations solve the optimal assignment problem and correspond one-to-one to competitive equilibrium allocations. Despite this result appearing a half-century ago and a growing body of recent research studying if decentralized individual behavior can reach core allocations, there is little evidence or explanation for when and how it fails and inefficient outcomes occur. In this paper, we formulate several models that represent strategic and bargaining game versions of interactions in such markets. The solutions to our basic models are generalizations of the core, and admit noncore allocations. These noncore allocations can be considered as predictions of "what can go wrong" in assignment markets. We then contrast the predictions of our strategic and bargaining models with the behavior of participants in laboratory experiments.

Our basic strategic model is a market game adapted from Dubey (1982), Simon (1984), and Benassy (1986). Buyers and sellers are allowed to bid simultaneously, with a clearing house picking an allocation consistent with the bid and ask prices submitted respectively by buyers and sellers. We show that every Nash equilibrium allocation is a competitive equilibrium allocation of an economy composed exclusively of the goods traded. That is, inefficiencies may arise due to coordination failures—buyers and sellers may fail to trade some goods that would be traded in an optimal assignment. In an alternative noncooperative game, following Pérez-Castrillo and Sotomayor (2017), we allow sellers to post prices before buyers are allowed to bid. Subgame perfection in this, now sequential, game selects the best competitive equilibrium for sellers, and therefore an optimal assignment.

Our basic bargaining model is a version of an asymmetric Nash bargaining solution that we call pairwise Nash bargaining (PNB). The solution to PNB consists of allocations that maximize the generalized Nash product among those that can be supported by pairwise transfers. These allocations may be suboptimal in the utilitarian sense, given the constraint to pairwise transfers. That is, they are inconsistent with competitive prices. We pay special attention to two allocations that can be supported by PNB, and that correspond to the point predictions of the leximin criterion, which embodies an egalitarian perspective, and the Shapley-Myerson value (after Myerson, 1977), which takes into account the pairwise structure of trade.

Our experimental treatments include continuous-time bidding, as is often the case in experimental markets, and in some cases free unstructured communication. We do not claim that our simplified models directly correspond to the markets induced in our experiment. Instead, we aim to contrast the predictions of our stylized models with behavior in realistic experimental markets. A faithful dynamic game-theoretic model would require either additional assumptions to make strong predictions, e.g. infinite experimentation assumption for stochastically stable sets, or a shift of the design away from a market experiment.[1] We are concerned rather with the question of finding a solution concept that both captures the overall market institution from the theoretical point of view, and predicts the outcomes well. Moreover, as argued later on, the outcomes predicted by our main models are also the sink points of probable convergent dynamic learning processes.

In the first treatment, we adapt a double auction as introduced by Smith (1962) and now common in market experiments with homogeneous goods (see e.g. the survey by Friedman (2018)). In the second treatment, we allow sellers to post prices before buyers bid for their goods, as in the sequential strategic game described above. In the third treatment, we allow for bilateral, private, free format communication between buyers and sellers, mimicking real-estate interactions, with the possibility of striking deals that traders can renege on by making another deal. In these three treatments, we do not inform traders of the valuations of other traders, and we repeat the experiment for several rounds with fixed roles and valuations, to allow for learning of market-relevant information and for experience. The fourth

[1] Newton and Sawa (2015) quote Boudreau (2012): "Calculating the probability of each stable outcome for a given market under the randomized tâtonnement process is extremely difficult due to the tremendous number of paths that can be involved." This claim has been contested in recent years with Newton and Sawa (2015) in particular offering an analytical solution to the nontransferable utility problem based on stochastic stability, Nax and Pradelski (2015) extending the approach to assignment markets, and Elliott and Nava (2019) offering a solution to the non-cooperative bargaining model in the same markets.

treatment offers a robustness check, and it is a version of the third treatment in which all participants have full information about valuations.

The possibility of bargaining simultaneously with multiple partners and reneging on agreements, the presence of search frictions as well as the choice of a structured trading or a bargaining protocol has been suggested before as a possible determinant of "who matches with whom" in such markets, especially in the experiments of Nalbantian and Schotter (1995) and Agranov and Elliott (2021). Based in part on this evidence, we would expect competitive pressure to be strongest in the first and second treatment, leading to outcomes consistent with the strategic model, while the freedom for renegotiation in the third treatment should lead to outcomes consistent with the bargaining model. We test if this connection holds up in practice by checking the accuracy of predictions in each case in the laboratory. The experimental protocol is kept compatible in all treatments, i.e. it is the market with the same induced supply and demand, but under different trading protocols.

Summarizing the experimental results, we indeed find bargaining models to be a relatively better fit for the treatments with private bilateral negotiations, while competitive prices and Nash equilibrium assignments occur relatively more often in the structured markets. As in other experiments,[2] communication favors coordination, leading to the optimal assignment, but also introduces either bargaining or pro-social behavior, in our case, exemplified by the frequency of the other matchings. Finally, the treatment where sellers move first favors them and allows them to reap some gains from the ability to commit. However, in spite of being induced by the unique subgame-perfect equilibrium, the optimal assignment is not reached in the posted price treatment as frequently as in the treatment with communication. Crucially, Nalbantian and Schotter (1995) document the same effects as our study with more matches overall in the free-form negotiation treatment, but a larger proportion of suboptimal matches, and vice-versa more unmatched players in an English auction. The present paper can be seen as proposing an explanation for these effects through the difference between bargaining and Nash equilibria.

[2]See e.g. Martinelli and Palfrey (2020).

The robustness treatment under complete information shows comparatively better performance of bargaining models, as expected. However, the complete information environment performs *worse* in terms of efficiency than the incomplete information counterpart, possibly because publicly known valuations shift the outcomes that subjects consider fair, thus leading to disagreement.

2. LITERATURE REVIEW

We separate theoretical and experimental literature and discuss the contribution and differences of our approach at the end of each subsection.

2.1. *Theory*

Related theoretical literature can be organized into three blocks: theoretical analysis of the core of the assignment game, strategic models including bargaining, and learning dynamics. We will cover these in this order.

From a theoretical perspective, assignment problems have been studied using linear programming techniques starting with Von Neumann and Morgenstern's (1953) discussion of exchange, motivated in part by Böhm-Bawerk's (1891) study of a competitive market. Gale (1960) proves the existence of competitive equilibria in assignment problems. The cooperative models of assignment markets offer axiomatizations and characterizations of competitive equilibria, but generally suggest the same set as a prediction—the core. Shapley and Shubik (1971) show that there is a one-to-one mapping from the core to competitive equilibria, which also possesses a lattice structure that orders the allocations from buyer-best to seller-best. These polytopes, "$45°$ lattices" in terms of Quint (1991), characterize exactly the set of all possible cores of assignment games. Ostroy (2018) shows the differences between the core and competitive equilibria in market games (and absence of this difference for assignment markets) through differentiability with respect to individuals and generators of positively homogeneous functions. A summary of results for assignment problems is offered by Roth and Sotomayor (1990, Chapter 8) and more recently by Núñez and Rafels (2015). Chade et al. (2017) connect the assignment and matching models to the literatures on sorting and search. The close connection between linear programming and competitive equilibria has also been explored recently in Baldwin and Klemperer (2019). Fair (envy-free) allocations in the assignment problem have been studied by Alkan et al. (1991) as a centralized problem under a budget constraint. Shapley and Scarf (1974) dispense with competitive prices to study the core itself and also use houses as an example. However, the lack of monetary valuations and transfers separates this study from Shapley and Shubik (1971) and also from the present paper, which is motivated by markets with monetary payments, such as real estate.

The second direction taken in the literature is to develop strategic models for assignment markets. While a powerful cooperative solution concept for the assignment problem with a non-empty set of predictions and an axiomatic characterization (see e.g. Toda, 2005), the core of the assignment game requires a particular market process to be implemented in a non-cooperative game. Schotter (1974) contains an early discussion of incentives for competitive behavior under different auctioning rules. Demange and Gale (1985) show that if a mechanism implements the optimal allocation for one side of the market, it may be manipulable by coalitions of players on the other side of the market. Pérez-Castrillo and Sotomayor (2002, 2017) provide sequential mechanisms where sellers announce their prices before buyers can bid for the goods, and show that such mechanisms implement the optimal allocation for sellers. If the game is taken as given, a possibility remains to make regularity assumptions to ensure that the core is exactly the whole set of predicted outcomes. Kamecke (1989) describes a version of non-cooperative characterization of an assignment market based on a demand game from Nash (1953), but imposes an exogenous cost of making high unfulfilled offers in the game to avoid noncompetitive Nash equilibria. Ott (2009) characterizes equilibria in a game of incomplete information where players participate in multiple second-price auctions simultaneously while also being interested in only buying one item. The results for auctions with small increments are similar to our simulations. A non-cooperative bargaining model on a graph that closely fits our setting is Elliott and Nava (2019), and they also review the literature on decentralized bargaining in markets.

A separate strand of literature on strategic models of assignment markets studies the convergence of learning dynamics in these games. A recent example applied to assignment markets and a short survey of similar results can be found in Pradelski and Nax (2019). The potential function in Demange et al. (1986) can also be viewed as evidence for the convergence of learning dynamics since it implies weak acyclicity for the game that implements the proposed one-sided mechanism.

In this paper we contribute to the strategic literature on assignment problems by studying the set of equilibria of market games. The equivalence result between (active trade) Nash and competitive equilibrium allocations obtained by Dubey (1982), Simon (1984), and Benassy (1986) does not hold in our setting because of market thinness, so characterizing the set of Nash equilibria is illuminating regarding possible outcomes of trading.[3] In this sense we move in the opposite direction from the theoretical literature that looks for restrictions on behavior or particular mechanisms like auctions or external subsidies to ensure that subjects always arrive at the core. To remain compatible with real-world markets where behavior may deviate from competitive outcomes and some markets may not open, we consider the market game to be given, i.e. our solution concepts are generalizations of the core, not refinements. To this end, we focus specifically on a market that does not exhibit supermodularity, and thus positive assortative matching does not immediately follow, making the search problem difficult for the players.

We contrast the derived set of Nash equilibria with predictions based on fairness and bargaining, including the leximin and versions of a Nash bargaining solution. The Pairwise Nash bargaining solution that we develop in the theory section is based on Myerson (1977) and Okada (2010), with payoffs restricted to be bargaining outcomes only within matched pairs. Nash bargaining models pose an additional problem in our environment due to the non-convexity of the feasible set. The details of extensions that remedy this are covered separately in Appendix C. The analysis would benefit from also checking the results against the predictions of the strategic bargaining model in Elliott and Nava (2019) for our

[3]Market games in the context of indivisible, homogenous goods have been considered by Friedman and Ostroy (1995) and Martinelli et al. (2019).

game. Unfortunately, the solution for our 3×3 experimental market appears to be both probabilistic, with multiple assignments occurring with positive probabilities, and difficult to compute.

2.2. *Experiments*

The heterogenous indivisible goods environment offers a larger spectrum of institutional possibilities than classic works on double auctions and pit trading markets, and it remains largely unexplored. The closest experimental studies to ours are Nalbantian and Schotter (1995) and Agranov and Elliott (2021). Both studies also compare an environment with free-form negotiations and a more structured treatment. The former experiment is conducted under incomplete information about trade surpluses, the latter with surplus matrix revealed to subjects. Similarly to this paper, Nalbantian and Schotter (1995) compare performance of an auction and free-form communication in a 3×3 scenario (this experiment also includes a centralized simultaneous mechanism). Agranov and Elliott (2021) instead compares a structured Rubinstein bargaining treatment with unstructured free-form bargaining, similarly combining a realistic unstructured experiment with a powerful stylized game-theoretic model, specifically a Markov Perfect equilibrium. Other studies of behavior in market settings with multiple goods and indivisibilities have largely been limited to centralized mechanisms, e.g. Rassenti et al. (1982). Experimental studies that mimic bilateral negotiations do this either through communication over the telephone (Hong and Plott, 1982, Grether and Plot, 1984), using private booths (Crössmann, 1982, Selten, 1970), or with subjects walking around the room as in the original experiment by Chamberlin (1948) (see also Plott, 1982). Unlike Selten (1970), we do not allow private communication between sellers or between buyers to conspire against the other side of the market. The comparisons between our treatments are also relevant to the discussion of recontracting in Edgeworth and Walrasian *tâtonnement* found in Walker (1973), with some experimental evidence for positive effect of opportunities for recontracting on efficiency (Smith et al., 1982).

The experimental part of the present paper can be viewed as an empirical test of convergence in markets for indivisible goods under realistic unfavorable conditions like lack of supermodularity, difficult search, and, optionally, imperfect information about trading opportunities. However, while theoretical studies of convergence of assignment markets like Nax and Pradelski (2015) focus on convergence to competitive outcomes, our Nash equilibria characterization suggests that markets can instead converge to a semi-Walrasian equilibrium, which is in line with experimental results. We also conduct simulations of several simple learning dynamics as a baseline.

At the same time, through our treatments we find evidence distinguishing behavior that is close to a bargaining model from behavior that is expected in a double auction environment. To illustrate this point further, we organize the rest of the related experimental studies of double auction and pit trading markets with both heterogeneous and homogeneous goods in terms of efficiency (Figure 1). We did not include the experimental designs that combine the two mechanisms into a centralized market with an option to trade off-floor (Campbell et al., 1991) or endogenous market structures where traders decide who will be informed about their offer (Kirchsteiger et al., 2005, directed bid-ask market treatment).

There is a common belief that auctions outperform decentralized bargaining in terms of the total surplus achieved by traders. The evidence for this here is mixed, but it is also suggested by within-study comparisons, e.g. in Kirchsteiger et al. (2005). Interestingly, our assignment markets and the thin experimental literature on bargaining and matchings with transferable utility (e.g. Nalbantian and Schotter, 1995, discussed above) point to the contrary—since renegotiation is now a valuable instrument for efficiency, pit trading treatment outperforms centralized market treatments. Moreover this is achieved, in line with previous bargaining experiments, through higher volume at more equalized prices. Of course this is only suggestive since there is no reasonable way to aggregate efficiency across studies because of the differences in trading protocols, gains from trade, the number of traders, and the complexity of finding the efficient matching.

The remainder of the paper is organized as follows. In Section 3 we first define the assignment game, the core and the competitive equilibrium. Next, we develop the non-

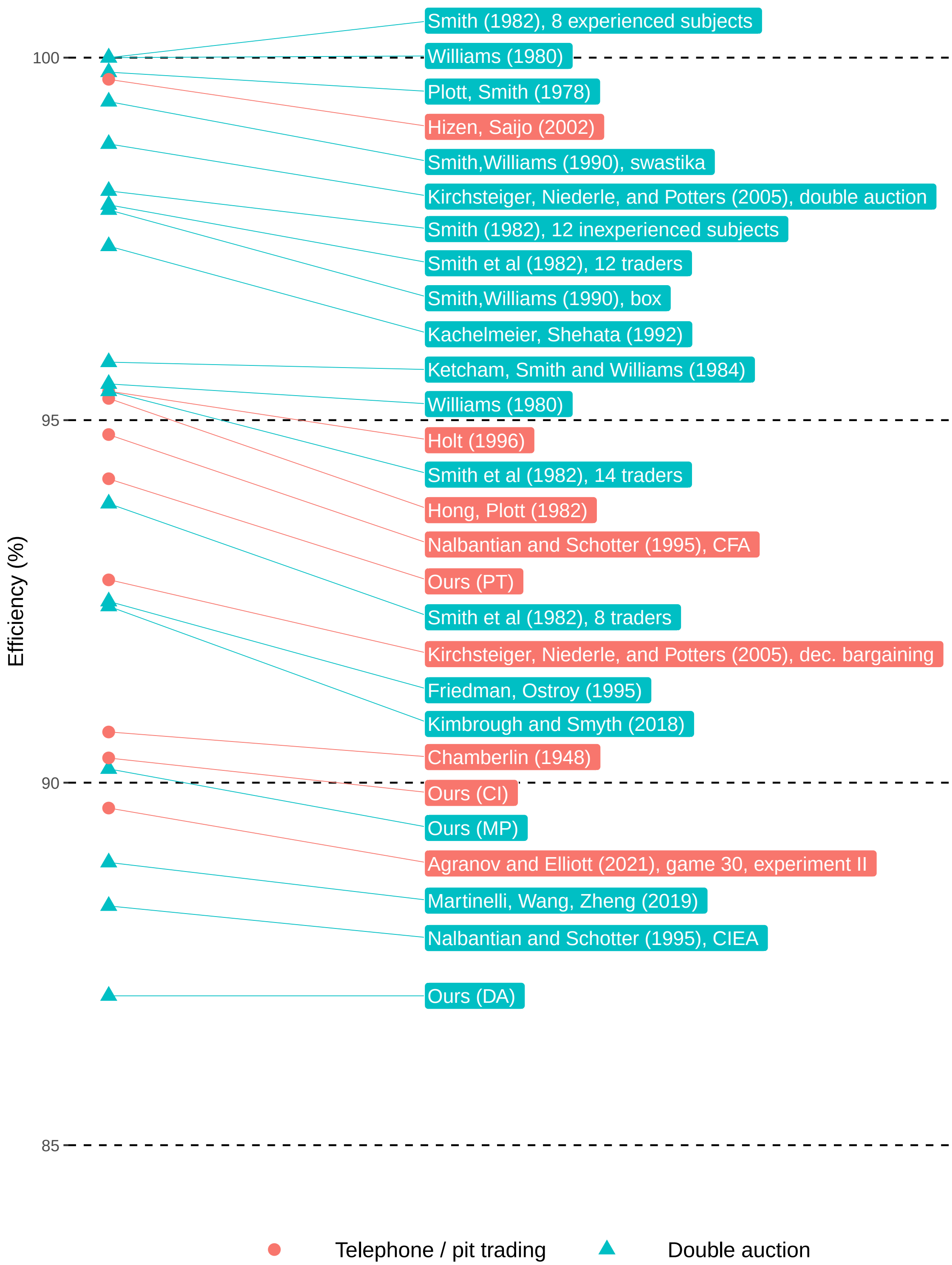


FIGURE 1.—Efficiency in the literature on double auctions and pit trading (telephone) negotiations

cooperative counterpart, the strategic market game, and characterize the Nash equilibria through competitive equilibria. We then do the same with bargaining models, starting with the Nash bargaining solution and then using it to develop our idea of a Pairwise Nash core. Finally, we discuss two remaining point predictions—the Shapley-Myerson value and the leximin. We conclude the section by showing the relationships between these concepts. In Section 4, we describe the experimental market, and use it to illustrate all the sets of outcomes predicted by different theories. In Section 5, we describe the experimental design and the four treatments. In Section 6 we provide results for simulated markets that offer a baseline for zero-intelligence or bounded rationality behavior. In Section 7, we relate theoretical predictions to experimental results. In Section 8 we discuss robustness. In Section 9, we conclude by outlining gaps and avenues for future research.

3. THEORY AND PREDICTIONS

3.1. *The assignment game*

Let B and S denote respectively the set of buyers and sellers, with $|B| = M$ and $|S| = N$. Each seller has one good to sell, and therefore we use S for the set of goods as well. Each buyer j has a valuation for each seller's good i, denoted by h_{ij}, while each seller has a reservation value for her own good c_i. Possible market operations are transfers of goods from a seller to a buyer, and transfer of money within a pair of trading partners.

An *assignment* of buyers to sellers can be represented by an $N \times M$ binary matrix x with $x_{ij} = 1$ whenever good i is allocated to buyer j, and $x_{ij} = 0$ for all j if the good i is not sold. The set of assignments is then:

$$X = \{(x_{11}, \dots, x_{NM}) : x_{ij} \in \{0, 1\}, \textstyle\sum_{j \in B} x_{ij} \leq 1 \text{ for all } i \in S\}.$$

Payoffs are linear in money; in particular, if buyer j buys a good from seller i and pays p_i. Every buyer is only interested in buying one good and her payoff is $\max_{i \in S}(x_{ij} h_{ij}) - \sum_{i \in S} p^B_{ij} x_{ij}$. The seller obtains payoff $p_i - c_i$. Let $a_{ij} \equiv \max(0, h_{ij} - c_i)$ be called the *trade surplus* of the buyer j and seller i. These surpluses form the $N \times M$ matrix A. We refer to (S, B) as an *economy*.

Following Aumann (1961), we define the core in terms of dominance, and also use this approach later for our bargaining concept. First, instead of the powerset on $B \cup S$, consider without loss of generality the set of valid trading pairs $\{(i,j) : i \in S, j \in B\}$, or $S \times B$ for short, called the *coalition structure*. Then a (set-valued) *characteristic function* assigns a set of pairs of payoffs to every coalition. The characteristic function v is defined as $v(\{i,j\}) = \{(u_i^S, u_j^B) \in \mathbb{R}_2^+ : u_i^S + u_j^B \leq a_{ij}\}$. In other words, it is the set of all payoffs attainable by the coalition.

The *feasible set* $U \subset \mathbb{R}_+^{M+N}$ is the set of payoff vectors or *imputations* that are attainable in some assignment. In particular, an $(M+N)$-payoff vector $(u) = (u^S, u^B)$ or simply $u = (u_1^S, \dots, u_M^S, u_1^B, \dots, u_N^B)$ is in the feasible set U if and only if there is $x \in X$ such that $u_i + v_j = h_{ij} - c_i$ if $x_{ij} = 1$, $u_i^S = 0$ if $\sum_{j \in B} x_{ij} = 0$, and $u_j^B = 0$ if $\sum_{i \in S} x_{ij} = 0$.

A payoff vector $\hat{u} \in R^N$ is said to *dominate* a payoff vector $u \in R^N$ with respect to characteristic function v if for some coalition $C = (i,j) \in S \times B$, there is $(u_i^S, u_j^B) \in v(C)$, such that $u_i^S < \hat{u}_i^S$ and $u_j^B < \hat{u}_j^B$. A *core imputation* is a feasible payoff vector $\hat{u} \in U$, that is not dominated by any other feasible vector.

The optimal assignment is obtained by solving the assignment problem:

$$\max_{x \in X} \sum_{i \in S} \sum_{j \in B} x_{ij} a_{ij} \text{ such that } \sum_{i \in S} x_{ij} \leq 1 \text{ for all } j \in B. \tag{3.1}$$

Shapley and Shubik (1971) show that the set of core imputations are solutions to the dual program:

$$\begin{gathered} \min \sum_{i \in S} \hat{u}_i^S + \sum_{j \in B} \hat{u}_j^B, \text{ such that} \\ \hat{u}_i^S + \hat{u}_j^B \geq a_{ij} \text{ for all } (i,j) \in S \times B, \text{ and } \hat{u}_i^S \geq 0, \hat{u}_j^B \geq 0. \end{gathered} \tag{3.2}$$

To describe competitive behavior, let $Y_j = \{0,1\}^N$, with typical element $y_j = (y_{ij})$, represent the set of possible demand vectors for buyer $j \in B$, with the interpretation that $y_{ij} = 1$ if buyer j demands good i and $y_{ij} = 0$ otherwise. That is, a buyer can acquire one good, several, or none. Similarly, let $Y_i = \{0,1\}$, with typical element y_i, be the set of possible supply decisions by seller i, with the interpretation that $y_i = 1$ if i sells her good and $y_i = 0$ otherwise.

A *competitive equilibrium* for the economy (S,B) is a pair (y,p), $y=((y_i)_{i\in S},(y_j)_{j\in B})\in\prod_{i\in S}Y_i\times\prod_{j\in B}Y_j$ and $p=(p_i)_{i\in S}\in\mathbb{R}^N_+$ such that

$$\max_{i\in S}(y_{ij}h_{ij})-\sum_{i\in S}p_iy_{ij}\geq\max_{i\in S}(y'_{ij}h_{ij})-\sum_{i\in S}p_iy'_{ij}\text{ for all }y'_j\in Y_j,\text{ for all }j\in B,$$

$$(p_i-c_i)y_i\geq(p_i-c_i)y'_i\text{ for all }y'_i\in Y_i,\text{ for all }i\in S,\text{ and}$$

$$\sum_{j\in B}y_{ij}=y_i\text{ for all }i\in S.$$

The first set of conditions represents utility maximization by buyers, and encode the assumption that buyers can enjoy at most one good. The second set of conditions represents profit maximization by sellers, and the third set of conditions are market clearing conditions for each of the goods. The set of core imputations coincides with the set of competitive equilibrium imputations.

3.2. *Strategic market game*

We can now define the strategic solution. In the strategic *market game* $\Gamma(S,B)$ each seller $i\in S$ submits a price $p^S_i\in\mathbb{R}_+$, and each buyer submits an N-vector of positive bids $p^B_j\in\mathbb{R}^M_+, p^B_j=(p^B_{1j},...,p^B_{Mj})$. The sets of admissible prices/bids for player $k\in B\cup S$ is denoted W_k. An *offer profile* combines actions of all players in a tuple $w=(p^B_1,\ldots,p^B_N,p^S_1,\ldots,p^S_M)\in W=\prod_{k\in B\cup S}W_k$.

Once all bids and prices are submitted, a clearing house chooses an assignment (or a lottery over assignments) from the set X to maximize surplus

$$\pi(x,w)=\sum_{i\in S}\sum_{j\in B}x_{ij}(p^B_{ij}-p^S_i),$$

i.e. it draws from the following set of surplus-maximizing assignments:

$$\bar{X}(w)=\{x\in X:\pi(x,w)\geq\pi(x',w)\text{ for all }x'\in X\}.$$

To ensure that the clearing house prefers more trade even when arbitrage is zero, we assume that it chooses assignments that are not ray-dominated (Simon, 1984):

$F(w)=\{x\in\bar{X}(w):\text{there is no }x'\in\bar{X}(w)\text{ such that}$

$$x' \neq x \text{ and } x'_{ij} \geq x_{ij} \text{ for all } i \in S, j \in B\}.$$

We also assume that the clearing house randomizes over the full $F(w)$, that is it chooses randomly according to some distribution that has positive probability on all $F(w)$.

Once the clearing house chooses an assignment $x \in F(w)$, the market clears at the buyers' prices,[4] and buyers and sellers get the payoffs

$$\max_{i \in S}(x_{ij}h_{ij}) - \textstyle\sum_{i \in S} p^B_{ij}x_{ij} \quad \text{and} \quad \textstyle\sum_{j \in B}(p^B_{ij} - c_i)x_{ij},$$

respectively.

Before we describe the set of Nash equilibria of the market game, note that for every subset $S' \subseteq S$ of sellers (including the empty set), we can define a smaller economy (S', B) with S' as the set of sellers and B as the set of buyers. Let (y', p') be a competitive equilibrium for any such economy with

$$y' = ((y'_i)_{i \in S'}, (y'_j)_{j \in B}) \in Y_{S',B} \equiv \textstyle\prod_{i \in S'} Y_i \times \prod_{j \in B} Y_j \quad \text{and} \quad p' = (p'_i)_{i \in S'} \in \mathbb{R}^{|S'|}_+.$$

With a slight abuse of notation, for any $y \in Y_{S',B}$, let

$$x(y) \equiv (x \in X : x_{ij} = y'_{ij} \text{ if } i \in S' \text{ and } j \in B, \text{ and } x_{i'j} = 0 \text{ if } i' \notin S' \text{ and } j \in B).$$

Note that if $x(y') \in F(w)$ for some bid profile $w = (p^B_1, \ldots, p^B_N, p^S_1, \ldots, p^S_M)$ such that

$$y'_{i'j} = 1 \quad \Rightarrow \quad p^S_i = p^B_{ij} = p'_i \quad \text{for every } i' \in S' \text{ and } j \in B,$$

then $F(w) = \{x(y')\}$, and moreover, for every $i' \in S'$ and $j \in B$ such that $y'_{i'j} = 1$, we have $p^S_i = p^B_{ij} = p'_i$. In other words, it induces the same assignment and the same prices for the goods in S', while the goods in $S \setminus S'$ are not sold. We will say in this case that a bid profile w "induces the same allocation" as the competitive equilibrium (y', p'). Note that w is a Nash equilibrium.

[4] Allocating surplus to one side of the market follows the convention in Dubey (1982) and also reflects the experimental treatment where one side of the market has the full market power.

The next two theorems characterize the Nash equilibria of this game and relate them to the competitive equilibrium.

THEOREM 3.1: *If* (y', p') *is a competitive equilibrium for some economy* (S', B) *for some* $S' \subseteq S$*, then there is a Nash equilibrium bid profile* w *that induces the same allocation.*

An immediate corollary of Theorem 3.1 is that every competitive equilibrium allocation of the original economy (and hence any corresponding core imputation) can be supported by a Nash equilibrium. However, other allocations can be supported as well; in fact, recalling that in our environment every good represents a different market, any arbitrary subset of markets can be shut down in a Nash equilibrium. This is the result of a coordination failure—intuitively, markets for particular goods may not open because each side of the market expects the other side not to show up. Such allocations are the semi-Walrasian allocations, which, as shown in Mas-Colell (1982), are more difficult to destabilize than other non-equilibrium allocations. That is, if some active market is not in an equilibrium, the allocation can be blocked by one type of traders ("1-blocking"), but if all active markets are in equilibrium, then at least m types of traders may be needed (m-blocking), with m no more than and sometimes exactly the number of inactive markets.

A converse result to theorem 3.1 also holds.[5]

THEOREM 3.2: *If* $F(w) = \{x\}$ *and* w *is a Nash equilibrium of a market game* $\Gamma(S, B)$*, then there is a competitive equilibrium* (y', p') *for some economy* (S', B)*, where* $S' \subseteq S$*, such that* x *induces the same allocation as* (y', p')*.*

The following corollary follows. Intuitively, if all markets open, the strategic game leads to a competitive equilibrium for the complete economy.

[5] Theorem 3.2 can be extended to Nash equilibria with random outcomes as long as the clearing house cannot react to an arbitrarily small price reduction by a buyer from an initial strategy profile w by increasing the probability of a different assignment which has the same surplus in the initial situation and excludes the buyer.

COROLLARY 1: *If w is a Nash equilibrium of $\Gamma(S, B)$ such that $F(w) = \{x\}$ satisfying $\sum_{j \in B} x_{ij} = 1$ for all $i \in S$, then x is a solution to problem 3.1.*

As an alternative noncooperative game, inspired by Pérez-Castrillo and Sotomayor (2002, 2017), consider a sequential game with complete information in which sellers are allowed to choose simultaneously their prices first, and then buyers choose simultaneously their bids, with the clearing house choosing the final allocation as before. By standard arguments, every Nash equilibrium of the simultaneous game corresponds to a Nash equilibrium of the sequential game in which the buyers choose the same bid no matter what happens in the first stage of the game. More interestingly, following Pérez-Castrillo and Sotomayor, there is a unique subgame-perfect equilibrium path and it leads to the best allocation for sellers in the core, corresponding to the imputation u^*.

3.3. *Nash Bargaining*

The other class of models that we consider are bargaining models, which are candidates for predicting behavior in treatments with communication.

We will first define the asymmetric Nash bargaining solution (NBS) for weights α, and sets of sellers and buyers S and B, denoted $b(\alpha, S, B)$ to be the set of solutions of the following optimization problem:

$$\max_{(u^S, u^B) \in U} \prod_{i \in S} (u_i^S)^{\alpha_i} \prod_{j \in B} (u_j^B)^{\alpha_j}, \tag{3.3}$$

where $\alpha \in \triangle_{M+N}$, the $(M+N)$-simplex, i.e. $\alpha_i, \alpha_j > 0$[6] and $\sum_{i \in S} \alpha_i + \sum_{j \in B} \alpha_j = 1$.

Note that for singleton S and B, this is a convex bilateral bargaining problem because the set U is convex, but this is not true generally.

Every outcome of asymmetric Nash bargaining is Pareto efficient, but there is a problem with using this as a solution concept for the non-convex assignment economy. The classic axiomatic Nash bargaining solution is not well-defined for nonconvex sets and there are

[6]Depending on the goal, both definitions are common—only strictly positive weights (Miyakawa, 2008), or with possible zero weights as well (Binmore et al., 1986).

several competing extensions. Depending on the choice among these extensions, the set of asymmetric Nash bargaining solutions is either a subset of Pareto optima or equals it.[7] The technical details can be found in Appendix C, but in all cases under free weights α bargaining solutions offer only a weak prediction. They also assume that all players bargain jointly over all outcomes. We are interested instead in a bargaining model that can both be applied to non-convex feasible sets and capture the pairwise nature of bargaining in an assignment market. We introduce one such model in the next section using the definition 3.3.

3.4. *Pairwise Nash core*

In this section we adapt the Nash bargaining and the core to describe players that bargain over items with a fair proportion in mind (bargaining power) under incentive compatibility. In practical terms, this is a stable allocation of a game where subjects are forming pairs with trading partners for a payoff determined by the asymmetric Nash bargaining solution.

Formally, the Pairwise Nash characteristic function v_α^{PN} for an assignment economy (S, B) with surplus matrix A and weight vector α is a function that assigns to each coalition $C = (i, j) \subset S \times B$ a pair of payoffs given by the pairwise Nash bargaining solution for i and j:

$$v_\alpha^{\text{PN}}(C) = (x_i, x_j) = b(\alpha, S, B).$$

Note that v_α^{PN} is completely independent of actions of other players unlike the definitions of the α-effectiveness (Aumann, 1961) and the Nash-effectiveness (Okada, 2010).

The Pairwise Nash (PN-)core of an assignment market A is the union of cores of A with respect to the family of Nash characteristic functions v_α^{PN} for all weights $\alpha \in \triangle_{M+N}$, i.e. the set of payoff vectors $u \in U$ that are not dominated by any other payoff vector in U with respect to v_α^{PN} for some α.

[7] The problem of approximating a non-convex Pareto frontier with solutions to a set of optimization problems, including the weighted Nash bargaining solution, is well known in the field of multi-objective optimization (Braun, 2018, Miettinen, 2012).

In practical terms, PN-core is just a formal way of capturing the idea that subjects are bargaining for a particular "fair" share of trade surplus when matched, and will break matchings looking for a better deal, knowing what the other players consider fair. As we will show below this simple model describes the data quite well.

PN-core resembles the Nash core in Okada (2010) but differs in several crucial ways. First, the Nash core relies on the convex set of mixed strategies. Since we do not see evidence for mixing, we limit players to pure actions, which in turn leads to a non-convex feasible set for coalitions larger than 2 players and therefore we cannot use the definition of Nash bargaining directly as in the case of Nash core. It also would imply a possibility of bargaining in groups larger than 2, which is not consistent with our intended communication structure and experimental design. Instead we limit the possible coalitions to pairs of sellers and buyers while restricting the payoffs to be consistent with Nash bargaining within each pair. Finally, the Nash core does not require the actions in a coalition to be a bargaining outcome—it instead requires the retaliation of the remaining players to be reached through bargaining, which is less relevant to the assignment market described here.

3.5. *L-core, Shapley-Myerson value and leximin*

Both the asymmetric Nash bargaining solution and the PN-core have very wide predictions. We will address this in the following sections by calculating the power and homogeneity of the empirically determined weights, but we can also consider several theories with narrow or point predictions.

The *least core (L-core)* corresponds to a stochastically stable set of a natural learning dynamic in an assignment game studied in Nax and Pradelski (2015), who show this fact by observing that this set is most robust to one-shot deviations. Formally, the *excesses* of players $i \in B$ and $j \in S$ in some outcome with prices p and payoffs u are

$$e_j^B = u_j^B - \max_{i \in S}(A_{ij} - u_i^S), \text{ and}$$

$$e_i^S = u_i^S - \max_{j \in B}(A_{ij} - u_j^B).$$

The minimal excess is then

$$e_{\min} = \min\left(\min_{i:\sum_{j\in B} x_i j > 0} e_i, \min_{j:\sum_{i\in S} x_i j>0} e_j \right),$$

and the L-core is the set of states that maximizes the minimum excess $e_{\min}$. In our market the L-core is the following set (with constants rounded to 2 digits): $p_1 = [433.33, 446.66], p_2 = 413.33, p_3 = 386.67$. L-core also generalizes the nucleolus.[8] That said it is a very strong prediction, a set of measure zero in our experimental setup. Consistently with our results for Nash equilibria, in the empirical section we report a relaxed version of L-core that is calculated only for the economy consisting of the goods that were traded.

Another attractive outcome to become a focal point for the subjects is the Shapley value (Shapley, 1953), which can also be motivated as a unique fair allocation rule and a totally stable outcome of the link formation game from Myerson (1977) (since the assignment game is superadditive). Without going into the details, if subjects were to choose with whom to communicate out of their potential trading partners, and picked an allocation that satisfies equity and efficiency conditions from Myerson (1977), they would arrive at the allocation given by the Shapley value.

There is a complication however because the Shapley-Myerson value is defined on the full coalition structure — all elements of the power set $2^{S\cup B}$. The assignment market traders cannot freely split payoffs between matched pairs, and we therefore would need to either allow lotteries or admit weakly Pareto-optimal solutions. We will calculate the Shapley value under the assumption that coalitions larger than 2 can split the surplus freely between members and then use the closest feasible point as a prediction.

Finally, let us consider the leximin solution defined over the set U. Leximin may be an attractive criterion for egalitarian traders; see e.g. Sen (1970). Leximin requires maximizing lexicographically the worst-off trader, then the second worst-off, etc. We can find the leximin solution by splitting surplus equally in every buyer-seller pair for every complete

[8]Nucleolus for the particular market in our experiment coincides with the kernel, see Nax and Pradelski (2015) that uses the same example market.

assignment, and then maximizing iteratively the worst off matched pair of traders and fixing their utility for the consecutive optimization until we get the payoffs for all traders.

For these strong theories we also calculate the distance to the corresponding regions as an alternative measure of predictive success.

3.6. *Relationships between solutions*

While the Nash equilibria correspond exactly to the semi-Walrasian equilibria in any assignment market, the relationship between the Nash bargaining solution, the core and Pareto optima is less straightforward, and is summarized in Figure 2. In particular, any core imputation can be supported by the asymmetric Nash bargaining solution, but not all Pareto optima generally have this property. In fact, the Pareto set is not convex, unless the problem is trivial, and all Pareto optima are attainable in the same assignment.

Figure 2 shows the inclusions in terms of payoff vectors between the main solution concepts described in this section. The additional set Z in this diagram is the subset of Nash equilibria where for any unmatched buyer and seller $i \in S$ and $j \in B$, we have $a_{ij} = 0$:

$$Z = \{u \in NE, \text{ and for any } (i,j) \in S \times B, m(i) = \emptyset \text{ and } m(j) = \emptyset \implies a_{ij} = 0\}.$$

The sets with a * correspond to the "internal" allocations in the core and Z:

$$Core^* = \{u \in Core, u \gg 0\} \quad \text{ and }$$

$$Z^* = \{u \in Z, u_k > 0 \text{ or } m(k) = \emptyset \text{ for all } k \in S \cup B\}.$$

These sets exclude the extreme allocations where some matched players get zero payoffs, which cannot be supported by a Nash bargaining solution with positive weights. The $Core^*$ set can also be empty if some players are always unmatched in the core, e.g. there are more sellers than buyers. The proofs for some of these inclusions are collected in Appendix B.

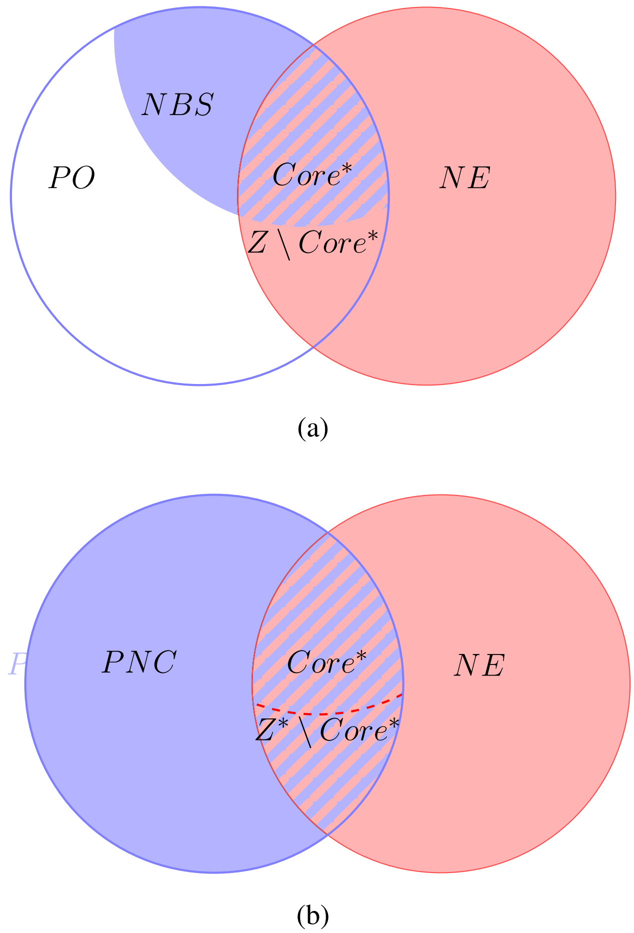


FIGURE 2.—Relationship between the Pareto Optima (PO), asymmetric Nash Bargaining Solution (NBS), Pairwise Nash Core (PN-Core), Core, and Nash equilibria (NE)

4. EXAMPLE AND THE EXPERIMENTAL ECONOMY

We will illustrate all predictions with an example borrowed from Shapley and Shubik (1971), which is described in Table I. This is also the economy used in all experimental treatments.

Every row in the table is a seller, and every column is a buyer. For this example, the values a_{ij} comprise the following matrix A, where each element is the joint maximal payoff of buyer j and seller i in experimental dollars (E$):

TABLE I

WIDGET VALUES FOR BUYERS AND SELLERS

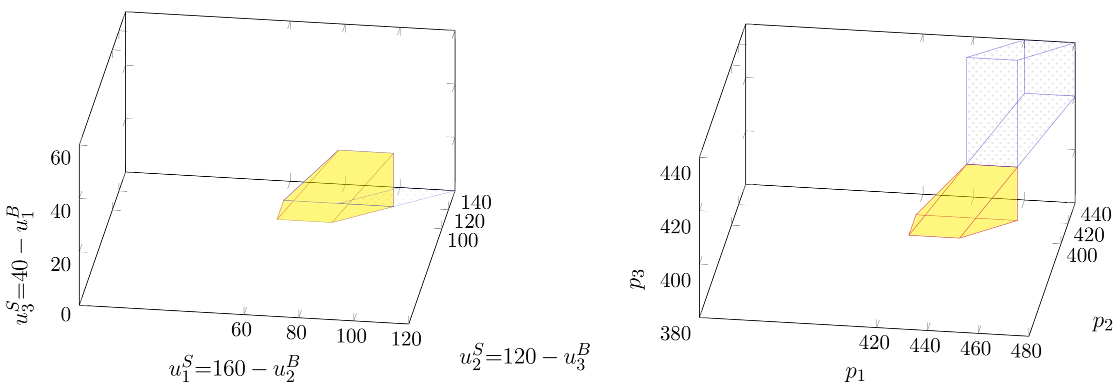
Widgets (i)	Seller's reservation value (E$) ($c_i$)	Buyers' valuations (E$) ($h_{i1}$)	(h_{i2})	(h_{i3})
1	360	460	520	400
2	300	440	480	420
3	380	420	440	340

$$A= \begin{array}{cc} & \text{(buyers)} \\ \text{(sellers)} & \begin{bmatrix} 100 & 160 & 40 \\ 140 & 180 & 120 \\ 40 & 60 & 0 \end{bmatrix} \end{array}.$$

The unique optimal assignment is obtained by solving problem 3.1, and is given by $x_{12} = x_{23} = x_{31} = 1$ and $x_{ij} = 0$ otherwise.[9] Optimal matches are shown circled in matrix A. We will use a shorter notation below, denoting assignments by three digit numbers. Each digit is the number of the good the buyer bought, e.g. the optimal assignment in the example can be written as $[312]$, where buyer 1 bought widget 3, buyer 2 bought widget 1 and buyer 3 bought widget 2. If a buyer does not buy anything, we will write zero.

Fixing the optimal assignment, we can find core imputations by using the constraints in problem 3.2. Since the constraints corresponding to optimal matches are binding, we have that all core imputations satisfy $u_1^S + u_2^B = 160$, $u_2^S + u_3^B = 120$, and $u_3^S + u_1^B = 40$. Projecting all core imputations into u^S space, we obtain the pentahedron depicted in Figure 3a. In particular, the buyer-optimal imputation is $(u^{S*}, u^{B*}) = ((60, 100, 0), (40, 100, 20))$ and the seller-optimal imputation is $(u^{S*}, u^{B*}) = ((100, 120, 20), (20, 60, 0))$. In between

[9]As pointed by Shapley and Shubik (1971), the optimal assignment is "normally" unique, as in the example below, in our experimental treatments and, generally, with probability 1 if the elements of the assignment matrix are drawn independently from a continuous distribution. There could be several optimal assignments in special cases, for instance if several goods are perfect substitutes for buyers.



a) Core imputations in sellers' payoffs space

b) Competitive equilibrium price vectors

FIGURE 3.—Core, Competitive equilibria and Nash equilibria

these two extremes lies the core of the game, which has a convenient structure. Let the partial order $u \leq_S u'$ capture seller-optimality, i.e. $u \leq_S u' \iff u_i^S \leq u_{i'}^S$ for all $i \in S$. Combined with this partial order, the core is a complete lattice.

Shapley and Shubik (1971) show that in any assignment game all core imputations $(\hat{u}, \hat{v})$ can be supported in competitive equilibria (y, p) by prices

$$p_i = \hat{u}_i^S + c_i \text{ for all } i \in S. \tag{4.1}$$

Conversely, all competitive equilibrium payoffs are core imputations. We therefore have a direct one-to-one mapping from core imputations to equilibrium price vectors,[10] as illustrated by a similar pentahedron in Figure 3b.

From corollary 1, the only complete assignment that can be induced (with probability one) by a Nash equilibrium is the optimal assignment $[\mathit{312}]$. Deleting rows in matrix A and solving 3.2, we get the other assignments that can be supported deterministically by Nash equilibria, ordered by total payoff gains: $[\mathit{210}]$, $[\mathit{320}]$, $[\mathit{310}]$, $[\mathit{020}]$, $[\mathit{010}]$, $[\mathit{030}]$, and

[10]The one-to-one relation described by equation 4.1 holds for widgets that are optimally assigned; for widgets that are not assigned to any buyer, any price $p_i \geq c_i$ is competitive.

$[000]$. The assignment $[210]$, shown circled in matrix A below,

$$A = f \begin{bmatrix} 100 & 160 & 40 \\ 140 & 180 & 120 \\ 40 & 60 & 0 \end{bmatrix},$$

is the closest to the optimal assignment by total payoff among all suboptimal assignments. This is the second most frequent assignment, after the optimal assignment, in the auction treatments of our experiments and it can be supported, for instance, by the following Nash equilibrium profile:

$$\begin{aligned} w &= (p_1^S, p_2^S, p_3^S, p_1^B, p_2^B, p_3^B) \\ &= (470, 430, 440, (460, 430, 380), (470, 430, 380), (400, 420, 380)). \end{aligned}$$

The prices in all Nash equilibria with assignment $[210]$ are shown as a blue dotted region in Figure 3. The price of the unsold third good has to be above a certain level as shown on the right plot, but the utility of seller 3 is zero as seen on the left.

The bipartite nature of the assignment market simplifies the analysis considerably compared to Mas-Colell (1982). Two players at the most need to change actions simultaneously to destabilize any Nash equilibrium. However, the transition from fixing the coordination problem in a single pair may affect many or all the agents. While we only need to change actions by the third buyer or the third seller in the assignment $[210]$ to nudge the markets toward the competitive equilibrium, the matching among active traders changes as well—buyer 1 is rematched with seller 3 instead of seller 2.

The third-best assignment $[132]$ in our example is the leximin and delivers a payoff of 30 to the two worst-off traders.

5. EXPERIMENTAL DESIGN

Our experimental sessions were conducted with 234 GMU undergraduate subjects in total, broken down into groups of six (three buyers and three sellers) for a total of thirty nine groups. Experiments were conducted in oTree (Chen et al., 2016). The three main treatments were held from September, 2018 to April, 2019 in person, and the fourth robustness treatment was conducted on-line in October–December 2020. The first 5 rounds were discarded as learning rounds, but this was not announced. These are dropped from all statistical analysis and tables. The software matched participants in groups of 3 buyers and 3 sellers with the matching and roles unchanged over the entire experiment. Subjects were recruited for 90 minutes. Earnings were calculated in the experimental currency and converted to U.S. dollars at the end of the experiment at a rate of E\$ 5 to US\$ 1. Only the earnings from one randomly chosen round were paid, with a mean payment of \$13.86 across treatments.

The treatments are motivated by the stylized interactions in the real markets. However, these games pose a theoretical problem if modeled faithfully as, for instance, a continuous-time dynamic game of bidding for the indivisible goods or a sequential bargaining procedure. The multitude of different equilibria arise by standard folk theorems, and these equilibria are often not stationary. Hendon and Tranaes (1991) show that markets as small as 1 seller and 2 buyers can have only non-stationary equilibria, and several of them. Instead, we study the outcomes predicted by classic static models of bargaining and competition and their stability. We do not focus on the bidding or bargaining processes that would allow traders to reach these outcomes within each round.

5.1. *Treatments*

In our experimental treatments, we inform traders of the number of other buyers and sellers in the group, as well as each trader's own parameters (e.g. the vector $(h_{ik})_{i\in S}$ if trader k is a buyer, and c_k if trader k is a seller), but, as in realistic market conditions, we do not inform them about the parameters of other traders. Thus, behavior under the different treatments can show how different market institutions promote information discovery by traders and induce competitive allocations, as those described in section 3.1, or instead leave traders stuck in suboptimal Nash allocations described in section 3.2, representing the failure of one or several markets to open. For each treatment we repeat the experiment with the same group for fifteen rounds to facilitate learning. The goods are called "widgets."

Buyers and sellers have induced valuations as in the previous section (Table I), which are constant between rounds and treatments. These payoffs follow the example in Shapley and Shubik (1971), but are scaled by a factor of 20 to form a large discrete space of integer bids and prices. The particular set-up in this table is attractive for the apparent complexity of the problem for the players, which can be seen through the simulations below.

5.2. *Double Auction (DA)*

The first treatment adapts the double auction commonly used in market experiments. In our version, the game is played in two stages. First, each seller sets a minimum price for her widget. This price has to be above her reservation value. When all minimum prices are set, the game proceeds to the trading stage. During the trading stage, buyers are allowed to bid for the widgets, and sellers are allowed to reduce their minimum prices. A buyer has a winning bid for a widget if her bid is the highest and it is above the current minimum price of the widget. To enforce unit demands, buyers who currently hold the winning bid for a widget are not permitted to make other bids until some other buyer outbids them.[11] The round ends after 50 seconds of inactivity and buyers with winning bids obtain their corresponding widgets. The game continues for fifteen rounds with groups, buyer and seller identities and reservation values unchanged between rounds, and with sellers revising their minimum prices at the first stage of each round. One round is then chosen at random for payment for each subject. The earnings for buyers are calculated as the difference between widget's value and the bid. Likewise, the earnings for the sellers equal the difference between the bid and the reservation value. Therefore sellers earn zero if they do not sell the widget and no trader is risking trading at a loss. Complete experimental instructions are available in the on-line appendix.

[11] In our strategic market game it is in principle feasible to allocate multiple non-trivial widgets to one buyer, but it can never be an equilibrium outcome. Experimental application in Ott (2009) demonstrates that subjects in a similar environment do not violate this assumption. That is, experimental subjects do not bid when they are the highest bidders for one of the items even if they are allowed to do so.

5.3. *Minimum Price (MP)*

The second treatment is set up to give market power to sellers, which should also help coordination. Like in the DA treatment, in the first stage each seller sets a minimum price for her widget. This price has to be above her reservation value, and this is the only way in which sellers actively participate in the market. When all minimum prices are set, the game proceeds to the trading stage with buyers bidding for the widgets.

This treatment is based on Pérez-Castrillo and Sotomayor (2002, 2017) and relates to job market matching models in Crawford and Knoer (1981) and Kelso, Jr. and Crawford (1982). It also mimics commonly observed posted prices for houses with subsequent negotiation phase. As in the sequential game of section 3.2, the commitment ability of sellers should shift the results toward seller-optimal core allocations (upper-right corners in both panels in figure 3).

5.4. *Pit Trading (PT)*

The third treatment is played through open-form bilateral communication between buyers and sellers. Every buyer has three chatboxes for private communication, one for each seller, and, similarly, each seller has three chatboxes, one for each buyer. Chatboxes contain two components, one for exchanging messages and one for negotiating the price. (Temporary) deals are marked in the chatbox shared by a buyer and a seller. Buyers and sellers can communicate in the three chatboxes simultaneously and back out of a deal at any moment, possibly striking another deal. The current negotiated deals are finalized when the round ends. Traders in the same side of the market can only communicate with the other side of the market and cannot communicate between themselves. This treatment is motivated by housing markets in which negotiations take place bilaterally over the phone or email. In this scenario, contracts are easier to back out from than in DA or MP.

5.5. *Pit Trading with Complete Information (CI)*

Finally, the fourth treatment was conducted as a robustness check. In this treatment all participants could see a table with costs and valuations (similarly to Table I) during trading turning the game into one of complete information and bringing the design closer to the theoretical model. This treatment was conducted on-line from October to December 2020 with the same subject pool (GMU students) and recruitment process, but with a higher show-up fee (US$ 10 against US$ 5 in the other treatments). The experimental design for this treatment is otherwise identical to the pit trading treatment.

6. SIMULATIONS

The simulations serve two reasons. The first is to adjust for power of Nash equilibria, bargaining and the rest of the models as predictors—some of these are of measure zero, some cover a large part of the feasible set, so a simple horse-race between the theories would not give correct results. The second reason is to compare the probability of converging to the prediction depending on the sophistication of the agents' strategies. We emulate three strategies: picking some better-response uniformly, picking a uniform better-response bid but always for a good with the highest value and price difference ("highest margin"), and the "almost best-responses," i.e. "highest margin" but with minimal price increments.

We emulate market dynamics by having subjects make random or boundedly rational decisions in random order, while maintaining all the restrictions of the experiment: not losing money, only increasing bids for buyers or reducing prices for sellers, and highest bidders not bidding for other goods. Because of these restrictions, this process may fall short of equilibrium or lead to miscoordination. In particular, any complete assignment that does not result in a negative utility is possible. Moreover, the zero in the third column of matrix A allows the third buyer and the third seller to be left unmatched in this process. If experimental subjects avoid these pitfalls this would suggest that they have foresight and learn not only within a trading round, but also between trading rounds.

The types of simulated dynamics are related to better and best-response dynamics in the literature on potentials and weak acyclicity—under best-response dynamics the price increments are always minimal and convergence is easier. Other types of dynamics have been shown to converge for such markets including "completely uncoupled" dynamics that do not require even knowledge about the structure of the game e.g. Nax and Pradelski (2015). In our case, since we frequently observe imputations outside of the core region, we are more interested in simple learning dynamics that allow subjects to consistently miss the core rather than the ones that lead only to core refinements. More importantly, since we focus on a strategic market game instead of the cooperative game formulation and the core, dynamics defined in terms of better and best-responses[12]) are the natural choice.

Predictions can be separated into statements about matchings and statements about prices and both are reported in Table 6 along with some descriptive statistics. We will use the same format for the experimental data in Table V for ease of comparison. The predictions about prices are stronger hypotheses as indicated by power analysis—the probability of efficient outcome $[312]$ is higher than the probability of competitive equilibrium or Nash equilibrium (competitive prices for traded goods). The probability of arriving at the efficient allocation with boundedly rational decisions is only $20.06 - 28.79\%$. It is much higher only if buyers behave near-optimaly with almost-best responses (81.94%) . Arriving at competitive equilibrium prices by randomizing is also unlikely, except under almost-best responses. We use arrows in Tables 6 and V to indicate nested theories, e.g. for any imputation $Core \implies PO$.

To provide an intuition for the high probability of the inefficient assignment $[\mathit{210}]$, note that a matching between buyer 1 and seller 2 cannot happen in a competitive equilibrium, because if $u_2 \leq 120$ and $v_1 \geq 20$ then seller 2 would want to sell to buyer 3 instead, and if $u_2 \geq 100$ and $v_1 \leq 40$ then buyer 1 would want to buy widget 3 instead. Under the rules of the Double Auction and Minimum Price treatments, if current prices indicate the first situation, buyer 3 will be able to outbid buyer 2 and the market would move toward the equilibrium. However in the second situation, buyer 2 cannot bid for good 3 since this buyer is already the highest bidder for good 2. Thus, the inability to renegotiate a deal may lead to inefficient assignments in the first two treatments.

We also conducted simulations for the Nash bargaining model and PN-Core. In our running example, the symmetric Nash bargaining solution with equal weights for all traders

[12]The process that we call "almost best-response" dynamics may fall short of equilibrium for two reasons: the highest bidders may have a profitable deviation but are prevented from changing goods by the bidding mechanism or the order of deviations does not lead to the Nash equilibrium. The convergence properties of assignment markets are studied in Dolgopolov and Martinelli (2021).

TABLE II

SIMULATIONS[a]

Assignment (% of observations)	Uniform	Highest margin	(Almost) best response
$[312]$ (Efficient)	20.06	28.79	81.94
$[210]$ (Unique 2^{nd} best)	20.09	24.95	2.09
$[120]$ (3^{rd} best)	19.33	21.83	3.97
$[321]$	15.28	12.42	3.36
$[132]$ (3^{rd} best and leximin)	13.48	6.88	7.40
$[231]$	11.76	5.12	1.25
Pareto Optimal (%)	94.67	96.27	99.60
↪ Pareto Optimal, not dominated by lotteries (%)	23.40	31.80	82.18
↪ Core = Competitive equilibrium = 0-blocked (%)	0.10	0.88	40.20
Competitive prices for traded goods (%)	20.59	18.50	55.90
↪ Least-core for traded goods (%)	0.02	0.032	0.006
Pairwise Nash core for some weights (%)	84.77	85.82	98.89
1-blocked (%)	1.18	3.63	22.43
2-blocked (%)	24.33	31.53	26.94
3-blocked (%)	70.12	61.54	10.37
$\geq$ 4-blocked (%)	4.28	2.42	0.06
Mean efficiency (% of maximum possible total payoff)	88.84	91.24	97.51
Mean seller's share of surplus (%)	81.20	81.05	51.99
Outcomes with 0-utility trades (% of observations)	15.23	14.18	1.11

[a] Based on 50000 simulated markets. Players make moves in random order. Buyers randomly pick a good if they can earn a positive payoff by buying it, sellers gradually reduce the minimum price. Only unmatched buyers can move. Uniform: buyer's bid increase is drawn from uniform distribution over all values that give positive payoff. Highest margin: buyers bid for goods that give them the highest payoff at current prices, but choose the new bid uniformly. Best response: same, but buyers only increment bids by minimum amount.

corresponds to the optimal assignment, but leads to the following price vector (transfers to sellers from matched buyers) $p = (440, 360, 400)$, which is not competitive. If we draw weights randomly, other (suboptimal) complete assignments will have positive probability, while incomplete assignments will have zero probability as long as weights are positive.

The outcomes of bargaining and Nash equilibria are thus notably different and if subjects were using learning models similar to these we would be able to distinguish between the two.

The PN-Core simulations (Table IV) show that not only all full assignments are possible for certain weight vectors, but also that $[210]$ (or the equivalent $[213]$) may happen for many weight combinations, i.e. the 0-surplus pair may remain unmatched. Overall, most theories are restrictive except for Pareto optimality and the Pairwise Nash core, which almost always fit the simulated datapoints.

TABLE III

GENERALIZED NASH BARGAINING FROM UNIFORMLY DRAWN WEIGHTS[a]

Assignment	Frequency
$[312]$ (Efficient)	58.8%
$[132]$ (3$^{\text{rd}}$ best and leximin)	35.1%
$[231]$	3.6%
$[321]$	2.5%

[a]Based on 1,000 draws with weights drawn from a uniform distribution.

TABLE IV

STABLE PN-CORE OUTCOMES FROM UNIFORMLY DRAWN WEIGHTS[a]

Assignment	Frequency
$[210]$ (2$^{\text{nd}}$ best, Nash)	30%
$[120]$ (3$^{\text{rd}}$ best)	27.8%
$[312]$ (Efficient)	24.2%
$[132]$ (3$^{\text{rd}}$ best and leximin)	9.2%
$[321]$	5.8%
$[231]$	3.0%

[a]Based on 50,000 draws with weights drawn from a uniform distribution.

7. ROBUSTNESS

One of the differences between the theoretical part and the experiment is the discrete set of options for bids in the experiment, while the theoretical model has continuous actions. The discrete action space is generally unavoidable since subjects are unlikely to bid irrational numbers. Importantly however, the discrete finite action space does not affect the set of Nash equilibria. It is easy to show that since the gaps between valuations of different goods are much higher than the smallest bid increment, the equilibrium matchings are unchanged, and equilibrium prices are within the minimal bid increment (1 in this experiment) from the Nash equilibrium.[13]

Unlike discrete bids, the choice of a private information setting for our experiments is non-trivial and has different implications for learning in our treatments. A natural critique would be that due to this private information, in addition to Nash equilibria in the market game, other outcomes may occur because subjects may hold incorrect beliefs about market opportunities. However the information structure is irrelevant for the repeatedly played strategic market game. Since subjects have private values, i.e. their payoff only depends on their value/cost and actions of other players (and not the other players' types), the best-responses too only require conjectures about the play of opponents. Thus beliefs about realizations of values/costs are irrelevant as long as observed actions are consistent with the conjectures as required in any self-confirming equilibrium. Formally, by Proposition 3 (ii) in Dekel et al. (2004), the set of such self-confirming equilibria reduces to the set of Nash equilibria of the complete information game with the realized payoffs that we study.

Moreover, while in other treatments there is room for bargaining between buyers and sellers, minimum price treatment is similar to a multiple auction setting in Ott (2009) and Demange et al. (1986). For this reason, even under incomplete information if subjects are only incrementing their bids in small amounts, a reasonable (although not dominant) strategy will lead them to an efficient assignment in a perfect Bayesian epsilon-equilibrium described in Proposition 3.6 in Ott (2009).

[13] This is proven in Dolgopolov and Martinelli (2021).

This may not be true for the bargaining environment that describes the PT and CI treatments as bidding is not fully observed and happens in private communication. This is why we conducted the robustness experiment CI above with the same subject pool and a similar protocol, but under publicly available cost and value information.

8. EXPERIMENTAL RESULTS

8.1. *Descriptive statistics and qualitative results*

Generally one may expect the DA and MP treatments to be closer to the predictions of the structured models—the Nash equilibria with competitive prices for the traded goods. On the contrary, the PT and CI treatments, due to the open communication nature and the possibility of renegotiation, are expected to be closer to bargaining models with outcomes around focal points like leximin and the Myerson-Shapley value or the predictions of PN-core and assignments in Table III. In this section we argue that this is indeed the case, starting with Table V that reports all main results, and then supporting the claims with statistical tests. We will also make frequent use of the large scatter plots of price vectors collected in Appendix 4.1.

First, in terms of assignments, the core assignment is the most or the second-most frequent assignment in all treatments. The second best Nash assignment $[210]$ is more frequent in DA, MP and CI, while the leximin $[132]$ is more frequent in PT. The assignment $[210]$ is precisely a semi-Walrasian equilibrium in terms of Mas-Colell (1982) and a Nash equilibrium in Theorem 3.2 with third market closed. Other Nash assignments are also more frequent in the structured DA and MP treatments. This is consistent with our expectations and with the results in Nalbantian and Schotter (1995), indicating that PT treatment solves the problem of unmatched players, but makes suboptimal matchings like leximin more probable.

We can see which effect dominates by looking at the overall efficiency reported with the overall statistics at the bottom of the table. The (incomplete information) PT treatment performs better than the other two with respect to efficiency. The rank-sum statistical test outlined in Table VI confirms this: PT advantage over the other two treatments is statistically significant. Efficiency under the PT treatment (94%) is also above efficiency of

TABLE V

FREQUENCIES OF ASSIGNMENTS ACROSS TREATMENTS (ROUNDS 6–15)

Assignment (% of observations)	DA	MP	PT	CI
$[312]$ (Efficient; Nash)	29	38	59	41
$[210]$ (Unique 2^{nd} best; Nash)	37	29	6	19
$[132]$ (3^{rd} best and leximin)	2	4	21	6
$[120]$ (3^{rd} best)	3	8	5	4
$[012]$ (3^{rd} best)	8	6	4	8
$[310]$ (Nash)	9	7	2	3
$[010]$ (Nash)	1	3	0	2
$[102]$	2	2	3	5
$[320]$ (Nash)	2	1	0	0
$[321]$	1	1	0	1
$[130]$	0	1	0	0
$[200]$	4	0	0	1
$[032]$	1	0	0	0
$[000]$ (Nash)	1	0	0	0
Pareto Optimal (%)	39	55	84	64.44
$\hookrightarrow$ Pareto Optimal, not dominated by lotteries (%)	29	39	59	45.56
$\hookrightarrow$ Core = Competitive equilibrium = 0-blocked (%)	16	24	10	8.89
Competitive prices for traded goods (%)	38	43	12	13.33
$\hookrightarrow$ Least-core for traded goods (%)	1	0	0	0
Pairwise Nash core for some weights (%)	60	60	84	74.44
1-blocked (%)	21	19	14	25.56
2-blocked (%)	29	28	56	36.67
3-blocked (%)	22	21	17	21.11
$\geq$ 4-blocked (%)	12	8	3	7.78
Efficiency (% of maximum possible total payoff)	87.06	90.19	94.19	90.34
Seller's share of surplus (%)	60.57	64.02	55.53	63.33
Outcomes with 0-utility trades (% of observations)	16	22	7	4.44
Total	100	100	100	90

"uniform" and "highest margin" random behavior (89% and 91%) from Table II, but below the "almost best-response" dynamics (97.5%). The fourth complete information treatment is close to the DA and MP treatments in terms of efficiency (90%), not the PT treatment that tests the same institution.

The frequent incomplete assignment also explains the higher sellers surplus in the MP, and DA treatments since the competition among the three buyers is higher for the two remaining goods with the competitive prices generally higher in the corresponding Nash equilibria. That this is indeed the effect can be checked by looking at the trading prices in the experiment, which fall in or close to the region predicted by the Nash equilibrium for the matching $[210]$ in DA and MP, but not in PT (the green region in D.1 in Appendix 4.1). The sellers also get the most of the surplus in CI treatment, but an inspection of the prices shows that they are significantly below the Nash equilibrium region for CI treatment and below a competitive price level. This supports our conjecture that the constraints of the competitive equilibrium are binding in the action environments, but not in the unstructured negotiation procedures.

There is an important caveat that, while the presence of the mismatched assignment $[132]$ is indeed evidence for bargaining and against Nash equilibrium outcomes, the lack of $[210]$ in PT by itself is not. The market in matrix A is a particularly complex problem because of a zero trade surplus between seller 3 and buyer 3, which makes the incomplete matching $[210]$ consistent with PN-Core as well.

Prices provide additional information for testing theories and the number of price vectors consistent with a particular theory is presented for each treatment in the second part of the table. While competitive prices are very rare under simulated behavior, they occur about a quarter of the time. As opposed to random behavior, and consistently with the Nash equilibrium, the frequency of the optimal assignment is related to the frequency of competitive prices and is also consistently higher in DA and MP. Another observation suggesting the bargaining nature of PT treatment is that prices arrived at by PT subjects do not group in regions where constraints bind as the DA and MP subjects do (see smaller panels in figure D.1 in Appendix 4.1).

TABLE VI

P-VALUES FOR RANK-SUM TESTS OF EFFICIENCY BETWEEN TREATMENTS

	DA	MP	PT	Remaining obs.
DA				0.0113^{**}
MP	0.2597			0.5599
PT	0.0011^{**}	0.0282^{**}		0.0018^{***}
CI	0.9886	0.2531	0.0237**	0.655

As to be expected from the design, the sellers' share is largest under the MP treatment, and the rank-sum statistical test outlined in Table VII confirms this, even though the subjects do not generally play according to the subgame-perfect equilibrium, i.e. the seller-best core allocation. The PT treatment is closer to equal split gains. The fourth complete information treatment shows surpluses similar to the DA and MP treatments, not the PT treatment.

TABLE VII

P-VALUES FOR RANK-SUM TESTS OF SELLER SURPLUS BETWEEN TREATMENTS

	DA	MP	PT	Remaining obs.
DA				0.1859
MP	0.1142			$<0.0001^{***}$
PT	0.0001^{***}	$<0.0001^{***}$		$<0.0001^{***}$
CI	0.2626	0.5852	$<0.0001^{***}$	0.0228^{**}

To further illustrate the price distribution under the different treatments, we calculate the average (minimum) Euclidian distance from observed price distributions to the core price region, conditional on the optimal assignment, the leximin assignment, and all assignments in Table VIII. Observed prices are closer to competitive prices under the DA and MP treatments conditional on reaching the optimal assignment, but not in general—the different

TABLE VIII

MEAN DISTANCE BETWEEN OBSERVED/SIMULATED DATA AND COMPETITIVE PRICES BY ASSIGNMENT.

	Observed data				Simulations		
Assignment	DA	MP	PT	CI	Uniform	Highest margin	(Almost) BR
Optimal	6.33	4.69	15.98	23.31	47.19	37.95	10.71
	(19.25)	(13.35)	(25.26)	(28.92)	(22.16)	(22.74)	(24.10)
Leximin	3.64	15.35	34.67	30.32	44.91	45.73	67.74
	(0.15)	(27.47)	(20.71)	(17.71)	(19.79)	(17.43)	(19.00)
All	20.93	19.06	22.09	23.31	45.47	39.67	18.33
	(35.62)	(36.32)	(26.76)	(28.92)	(19.26)	(20.17)	(29.22)

equilibrium prices under other assignments appear to drive the prices away from the core region in DA and MP treatments and make them almost as distant as in PT and CI. Conditional on the leximin assignment, observed prices are much further away under the PT treatment; values conditional on the leximin assignment for the DA and MP treatment are included for completeness though the leximin assignment is fairly rare for those two treatments. Compared to simulations, experimental subjects reach the competitive prices easier than simple subjects, but more rarely than (almost) best-responding subjects.

Calculating distances allows us also to see how close the datapoints are to the Myerson-Shapley value, L-core and the leximin price vector, the restrictive theories or predictions outside of the feasible set (Table IX). The L-core and Myerson-Shapley value are calculated for the economy consisting of only the traded goods for any given datapoint. For the infeasible Myerson-Shapley payoff vector we first find the closest feasible price vector. Even though the real subjects are generally closer to these points than simulated subjects, we can see that the points are further away from all these predictors and are closer to the core region.

For completeness we can also test the core refinements among the points that reach the optimal core assignment $[312]$ (not all of these points are in the core in terms of prices). These distances are reported in Table X. We can see that the points do not concentrate at the extremes and are closer to the middle core allocations in the L-core even for the

TABLE IX

OTHER THEORIES: MEAN DISTANCES TO MYERSON-SHAPLEY VALUE, LEXIMIN AND THE CORE.

	Observed data				Simulations		
min. distance to	DA	MP	PT	CI	Uniform	Highest margin	(Almost) BR
Myerson-Shapley	26.25	28.30	32.84	27.86	65.04	61.91	32.66
	(14.07)	(18.43)	(20.98)	(19.57)	(18.08)	(18.91)	(26.39)
Leximin	66.00	69.85	55.15	53.51	91.58	95.04	57.81
	(10.72)	(19.02)	(12.93)	(11.59)	(26.27)	(23.27)	(17.01)
L-core	30.25	27.24	35.05	40.34	51.66	46.13	34.45
	(23.72)	(20.36)	(28.80)	(30.64)	(20.96)	(21.60)	(28.29)

TABLE X

MEAN DISTANCES TO CORE REFINEMENTS (CONDITIONAL ON THE OPTIMAL ASSIGNMENT)

	Observed data				Simulations		
min. distance to	DA	MP	PT	CI	Uniform	Highest margin	(Almost) BR
Seller-optimum	29.73	34.36	43.43	36.13	50.40	41.55	51.31
	(20.17)	(13.60)	(22.94)	(17.52)	(22.14)	(21.77)	(19.65)
Buyer-optimum	30.88	24.08	29.82	25.16	78.39	72.80	16.73
	(17.29)	(21.44)	(23.35)	(15.33)	(22.02)	(22.95)	(27.58)
L-core	15.43	15.56	24.22	19.14	58.60	50.42	26.73
	(20.04)	(14.74)	(26.65)	(17.61)	(21.05)	(22.31)	(22.83)

MP treatment that favors sellers. The distances are also smaller than for simulated subjects, which tend to concentrate at the extremes. Thus, conditional on optimal assignment, L-core appears to be a good prediction with subjects tending to choose the allocations that do not favor either side.

We conclude the description of the data with the discussion of PN-Core. The PN-Core requires special attention to avoid overfitting—the theory has very low power when weights are free even though any given vector of weights leads to one specific assignment as can be seen in simulations. We can check if we indeed require significant heterogeneity in weights to explain the data by minimizing the variance in empirically determined bargaining weights. The standard deviations of the weights among the subjects consistent with PN-Core is 0.1, 0.08, 0.13, 0.09 for DA, MP, PT and CI respectively. It is perhaps easier to interpret these values in terms of payoffs—the difference in the potential trade surpluses from negotiating within any pair of subjects is within 7%, 18%, 9%, 13% from the mean in the whole sample of subjects consistent with PN-Core for DA, MP, PT and CI respectively. In other words all of the variance in prices and assignments among these subjects can be explained by a relatively slow variation in what they consider fair, the split of the surplus that they are bargaining for.

8.2. *Predictive Success and Market Dynamics*

In all statistical comparisons, we treat rounds as independent observations. This assumption rarely holds in practice for market experiments, and indeed in our case, the more common outcomes tend to persist from round to round (see also the Figure 4). This, however, does not seem to affect the intuition behind the results. On the contrary - the comparisons for simple dynamic models lead to similar conclusions. In particular, if instead of assuming independence, we considered every market to be a simple Markov chain that transitions between different assignments with some unknown probability, we could statistically compare the observed transitions. Specifically, the empirical probability of renegotiating the efficient assignment again after reaching it in the previous round is 41% for MP, 36% for DA, 85% for PT, and 66% for CI treatment (differences of MP and DA with PT significant at $p < 0.01$ according to Clopper-Pearson confidence intervals, but not with CI). The four tables with transitions between assignments for each treatment are shown in more detail in Figure 5. The three most frequent matchings discussed above—$[\mathit{312}]$, $[\mathit{210}]$, and $[\mathit{132}]$—prove to also be persistent with transitions between the first two describing most of the dynamic for the DA, MP and CI treatments, while the leximin $[\mathit{132}]$ and the efficient $[\mathit{312}]$ prove to be stable sinks for PT treatment.

The stability of assignments in PT and CI can perhaps be attributed to the stability of bargaining weights as an exogenous attribute, which leads to the same allocation across rounds, while both Nash equilibria are transient effects of price discovery as subjects move between them. A simpler behavioral explanation is that the personal nature of negotiations leads subjects to stick with their partners across rounds, but this is not consistent with the CI treatment where the protocol is the same as PT.

The random simulations that we conducted also allow us to calculate the Predictive Success Index (PSI, Selten (1991)), a measure that adjusts the frequency of successful predictions by the power of the test. The values of PSI for the four treatments can be obtained

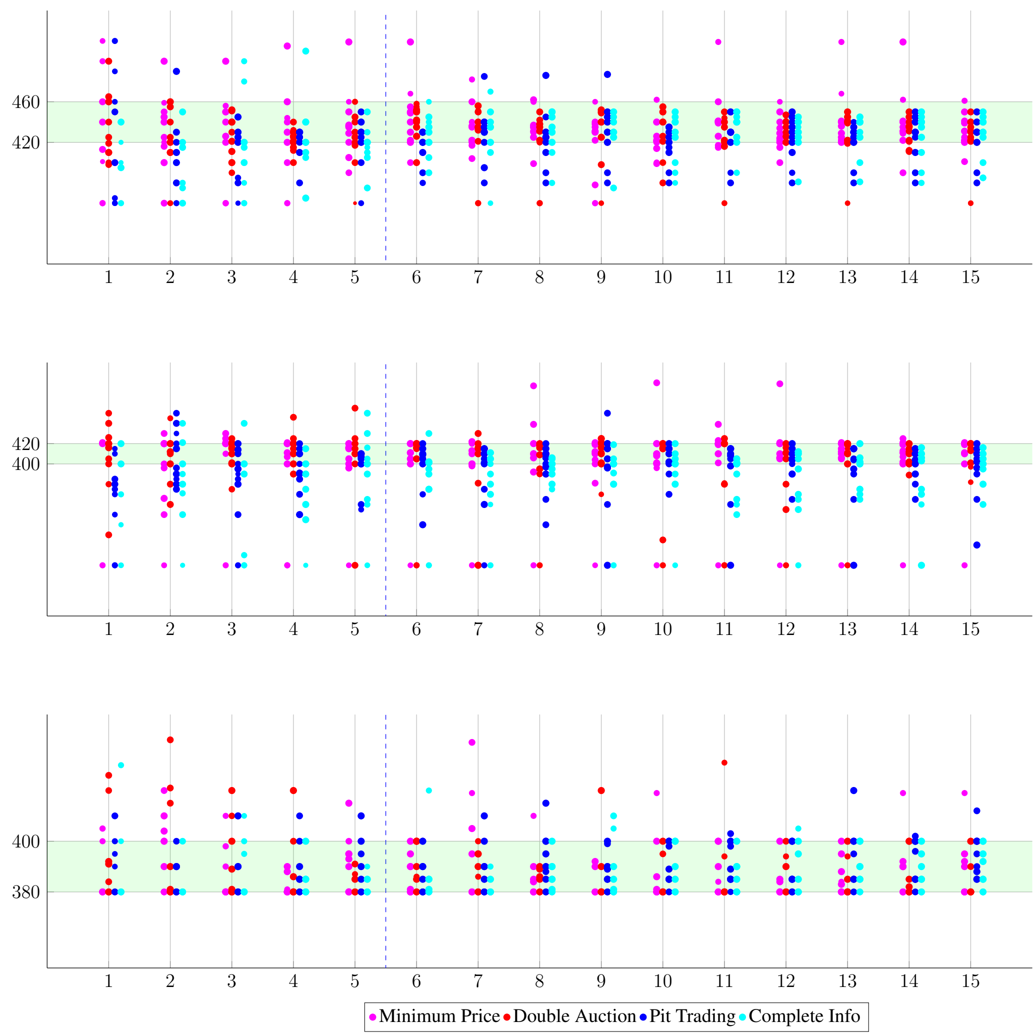


FIGURE 4.—Resulting prices across rounds[a]

[a]Light green band is the range of core prices.

	000	010	012	032	102	120	130	132	200	210	310	312	320	321
000	0	0	0	0	0	0	0	0	0	1	0	0	0	0
010	0	0	1	0	0	0	0	0	0	0	0	0	0	0
012	0	0	2	0	0	1	0	0	0	4	0	0	0	0
032	0	0	0	0	0	1	0	0	0	0	0	0	0	0
102	0	0	1	0	0	0	0	0	0	0	1	0	0	0
120	0	0	0	0	0	0	0	0	1	1	0	1	0	0
130	0	0	0	0	0	0	0	0	0	0	0	0	0	0
132	0	0	0	0	0	0	0	0	0	1	0	1	0	0
200	0	0	0	0	0	0	0	1	0	1	0	1	1	0
210	1	1	1	1	1	0	0	0	0	15	3	11	0	1
310	0	0	0	0	0	0	0	0	0	2	4	1	0	0
312	0	0	2	0	1	1	0	0	1	7	1	10	1	0
320	0	0	0	0	0	0	0	0	0	0	0	2	0	0
321	0	0	0	0	0	0	0	0	0	0	0	1	0	0

(a) Double Auction (DA)

	000	010	012	032	102	120	130	132	200	210	310	312	320	321
000	0	0	0	0	0	0	0	0	0	0	0	0	0	0
010	0	0	0	0	0	0	1	0	0	1	0	0	0	0
012	0	0	0	0	0	1	0	0	0	1	0	4	0	0
032	0	0	0	0	0	0	0	0	0	0	0	0	0	0
102	0	0	0	0	0	0	0	0	0	1	0	1	0	0
120	0	0	1	0	0	0	0	1	0	0	1	4	0	1
130	0	0	0	0	0	0	0	0	0	1	0	0	0	0
132	0	0	0	0	0	1	0	1	0	0	0	1	1	0
200	0	0	0	0	0	0	0	0	0	0	0	0	0	0
210	0	0	3	0	2	0	0	0	0	9	3	10	0	0
310	0	3	0	0	0	2	0	0	0	1	1	0	0	0
312	0	0	2	0	0	2	0	1	0	12	1	14	0	0
320	0	0	0	0	0	0	0	1	0	0	0	0	0	0
321	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(b) Minimum Price (MP)

	000	010	012	032	102	120	130	132	200	210	310	312	320	321
000	0	0	0	0	0	0	0	0	0	0	0	0	0	0
010	0	0	0	0	0	0	0	0	0	0	0	0	0	0
012	0	0	0	0	0	0	0	1	0	0	0	3	0	0
032	0	0	0	0	0	0	0	0	0	0	0	0	0	0
102	0	0	0	0	1	1	0	0	0	0	0	0	0	0
120	0	0	0	0	0	1	0	2	0	0	0	0	0	0
130	0	0	0	0	0	0	0	0	0	0	0	0	0	0
132	0	0	0	0	2	1	0	14	0	0	0	3	0	0
200	0	0	0	0	0	0	0	0	0	0	0	0	0	0
210	0	0	0	0	0	1	0	0	0	2	0	1	0	0
310	0	0	1	0	0	0	0	0	0	0	0	1	0	0
312	0	0	3	0	0	1	0	1	0	4	2	44	0	0
320	0	0	0	0	0	0	0	0	0	0	0	0	0	0
321	0	0	0	0	0	0	0	0	0	0	0	0	0	0

(c) Pit Trading (PT)

	000	010	012	032	102	120	130	132	200	210	310	312	320	321
000	0	0	0	0	0	0	0	0	0	0	0	0	0	0
010	0	0	0	0	0	0	0	1	0	0	0	0	0	0
012	0	0	1	0	2	0	0	0	0	2	0	2	0	0
032	0	0	0	0	0	0	0	0	0	0	0	0	0	0
102	0	1	1	0	0	0	0	1	1	1	0	0	0	0
120	0	0	0	0	0	1	0	0	0	2	1	1	0	0
130	0	0	0	0	0	0	0	0	0	0	0	0	0	0
132	0	0	0	0	1	2	0	0	0	0	0	0	0	0
200	0	0	1	0	0	0	0	0	0	0	0	0	0	0
210	0	1	3	0	0	2	0	0	0	3	1	7	0	0
310	0	0	0	0	0	1	0	0	0	1	0	1	0	0
312	0	0	0	0	2	0	0	1	0	9	2	23	0	1
320	0	0	0	0	0	0	0	0	0	0	0	0	0	0
321	0	0	0	0	0	0	0	0	0	0	0	1	0	0

(d) Complete Information Pit Trading (CI)

FIGURE 5.—Transitions between matchings (total incidence per treatment)

by subtracting values for simulated power tests in Table II from actual results for human subjects in Table V. Depending on which of the columns in Table II we choose for this, the interpretation of the PSI differs. The uniform simulations allow us to adjust for the approximate area of the region consistent with a given theory, e.g. Pareto optimal region is much larger than the core, which makes it easy to arrive at Pareto optimal outcomes, but it does not necessarily make it a better theory.

If instead we use "almost best-responses" we anticipate sophisticated subjects and consequently downplay the predictive power of most of the theories—it is easy to reach the core if one adapts behavior almost optimally. In this case we are looking for a theory that captures human behavior that is visibly different from the adaptive process of "almost best-responses" and not necessarily for the better. In other words this PSI indicates how much real human behavior, driven by fairness, bounded rationality and similar concerns, deviates from the outcomes that would be reached by almost optimal adaptive play.

Predictive success for all theories and both uniform and sophisticated simulated subjects baselines are presented in Table XI. We start with the uniform case in the top part. Once again, competitive prices yield good predictions for DA and MP treatments, while Pareto Optimality can generally be reached by human subjects in PT and CI. Consistent with expectations, for PT and CI treatments a less restrictive prediction of efficient matchings performs better when power is taken into account. That is, full efficiency is reached more often, but not necessarily through the competitive price mechanism. The table also reveals that the CI treatment is very poorly described by the competitive equilibrium despite showing similar assignments to the treatments with the centralized trade mechanisms. Indeed as we already know from studying the price vectors, the CI prices are significantly lower than expected in a competitive equilibrium.

For more sophisticated simulated agents playing "almost best responses" PSI is negative, which is to be expected—convergence is much easier in this case, and the interpretation of PSI is rather how much noise does human behavior introduce into this convergence process—the further the subjects' behavior deviates from the theoretical convergent dynamic, the further is the negative PSI from zero. For example, if the subjects were to behave according to our "almost best-response" dynamics they should have converged to the core about 30.2% more often than in the data for PT treatment. Only the PN-core is near zero for PT treatment and the second-highest value in the table, but negative for CI. In our case this means that both simulations and experimental subjects generally reach the outcomes consistent with PN-Core with very high probability. The overall interpretation of the bottom part of Table XI is that either subjects play near-optimally according to the game implied by the PN-Core, or they are playing according to the competitive forces of the Nash equilibria and the core, but are plagued by biases that drive the outcomes away from what would be expected.

We can formalize a statistical test for this table using the properties of variance of difference between two binomial proportions. By definition, any PSI is a difference between binomial proportions of subjects and randomly simulated points consistent with a theory. We use the MKInfer R package (Kohl, 2020) to calculate the confidence intervals for PSI, which are shown in the table.[14] We can almost always reject the hypothesis that PSI is the same between the PT treatment and the auction treatments DA and MP - the confidence intervals only cross for the core. Comparing CI with MP and DA, we can also reject the null for Nash equilibria (competitive prices for traded goods). The rest of the theories perform much worse in CI treatment with less of the efficient matchings or Pareto optimal outcomes.

9. CONCLUSION

The inability of the core concept to capture bargaining and strategic aspects of trading was noted both by Shapley and Shubik (1971) and Von Neumann and Morgenstern (1953), in a critique of the classic Böhm-Bawerk (1891) solution for the horse market assignment problem. As noted by Shapley and Shubik (1971), the choice of the appropriate solution concept is dictated by the institutional form, including the communication structure. This logic is evident in the experiment.

[14]There is considerable literature comparing different approaches to calculating confidence intervals for a difference of two proportions. See also Altman et al. (2013).

TABLE XI

PREDICTIVE SUCCESS

Uniform				
Theory	DA	MP	PT	CI
Pareto Optimal	-55.67	-39.67	-10.67	-30.22
	$(-64.65, -45.87)$	$(-49.42, -30.28)$	$(-19.09, -4.76)$	$(-40.52, -21.11)$
↪ Pareto Optimal under lotteries	5.60	15.60	35.60	22.16
	$(-2.39, 15.15)$	$(6.61, 25.41)$	$(25.80, 44.74)$	$(12.26, 32.43)$
↪ Efficient matching	8.94	17.94	38.94	25.49
	$(0.95, 18.48)$	$(9.03, 27.73)$	$(29.13, 48.08)$	$(15.59, 35.76)$
↪ Core = Competitive equilibrium	15.90	23.90	9.90	8.79
	$(10.00, 24.32)$	$(16.60, 33.14)$	$(5.43, 17.34)$	$(4.48, 16.47)$
Competitive prices for traded goods	17.41	22.41	-8.59	-7.25
	$(8.50, 27.21)$	$(13.14, 32.20)$	$(-13.60, -0.77)$	$(-12.80, 1.29)$
Pairwise Nash core	-25.82	-25.82	-1.82	-11.38
	$(-35.62, -16.75)$	$(-35.62, -16.75)$	$(-10.25, 4.09)$	$(-21.26, -3.49)$
(Almost) Best response				
Theory	DA	MP	PT	CI
Pareto Optimal	-60.60	-44.60	-15.60	-35.16
	$(-69.59, -50.80)$	$(-54.36, -35.22)$	$(-24.02, -9.70)$	$(-45.45, -26.05)$
↪ Pareto Optimal under lotteries	-53.18	-43.18	-23.18	-36.62
	$(-61.17, -43.63)$	$(-52.17, -33.37)$	$(-32.98, -14.04)$	$(-46.52, -26.36)$
↪ Efficient matching	-52.94	-43.94	-22.94	-36.38
	$(-60.93, -43.39)$	$(-52.84, -34.14)$	$(-32.74, -13.80)$	$(-46.28, -26.12)$
↪ Core = Competitive equilibrium	-24.20	-16.20	-30.20	-31.31
	$(-30.12, -15.76)$	$(-23.52, -6.95)$	$(-34.69, -22.75)$	$(-35.64, -23.61)$
Competitive prices for traded goods	-17.89	-12.89	-43.89	-42.56
	$(-26.80, -8.09)$	$(-22.17, -3.10)$	$(-48.91, -36.07)$	$(-48.11, -34.00)$
Pairwise Nash core	-24.77	-24.77	-0.77	-10.32
	$(-34.57, -15.70)$	$(-34.57, -15.70)$	$(-9.19, 5.15)$	$(-20.21, -2.44)$

Our experimental results notably follow the 70 years of intuition in experimental markets with negotiation—as first noted by Chamberlin (1948), the volume of trade in such “imperfect” markets is above the equilibrium level, and the prices are below the equilibrium level. Our experiment suggests that these effects extend to heterogeneous goods. We also observe a larger volume of trade and lower seller surplus when comparing the pit trading treatment with the other treatments. The higher volume of trade is consistent with the explanations based on imperfections in the matching procedure, which in our case also helps avoid the miscoordination of the semi-Walrasian outcomes. The usual explanations for the lower prices are behavioral—the buyer’s perception or fairness of seller’s prices is also discussed as early as Chamberlin (1948). We complement these explanations with a bargaining model in the heterogeneous good setting that has a similar intuition and fits the data.

A higher efficiency in the treatment with unstructured negotiations could be taken as a suggestive explanation of why open non-centralized negotiations are prevalent in real estate, with the proviso that the opportunities for communication lead to trades at prices that differ from competitive prices, a possibility that indeed was implicitly entertained by both Von Neumann and Morgenstern (1953, p. 564) and Shapley and Shubik (1971, p. 128).

While the characterization of Nash equilibria is a simple generalization of competitive equilibrium, bargaining models are difficult to adapt to assignment markets. There is room for future research in adapting non-convex Nash bargaining extensions and axiomatic concepts like the Myerson value to this setting while maintaining strong Pareto optimality. The Myerson value, in particular, is a very general solution concept, and if we were to allow free disposal instead of lotteries, we could have a prediction within the feasible set (following the more general definition offered in the last part of Myerson (1977)). However the result would be only weakly Pareto optimal—it would imply that a matched pair of a buyer and a seller are burning their surplus out of concern for other players with poorer trading opportunities. While it may be a useful idea for another scenario, it is not consistent with the logic of an assignment game, in which a buyer and a seller would likely want to split the pairwise surplus. An interesting avenue would be to extend the Myerson value to non-convex problems like assignment markets without adding weakly Pareto optimal points. It is also promising to extend the Nash equilibrium characterization and the experiment to more involved markets like Fleiner et al. (2019) and Bikhchandani and Ostroy (2002).

Finally, the detrimental behavioral effect of complete information evident in the CI treatment is of interest and could be explained by a fairness motive that leads subjects to disagree more often. For the sake of full disclosure it should be noted that, unlike the rest of the analysis, this is an ex-post explanation for an effect that we did not fully anticipate or planned to test.

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APPENDIX A: THEOREM PROOFS

PROOF OF THEOREM 3.1: To prove the theorem, consider w such that

$$p_i^S = \begin{cases} p_i' \text{ if } i \in S' \\ \kappa \text{ if } i \notin S' \end{cases}$$

for some $\kappa > \max_{i \in S, j \in B} h_{ij}$, and

$$p_{ij}^B = \begin{cases} p_i' \text{ if } y_{ij}' = 1 \\ 0 \text{ otherwise} \end{cases}.$$

While the assignment $x(y')$ makes zero surplus, every other assignment makes zero or negative surplus, and moreover the assignment above ray-dominates every other zero-surplus assignment. Hence, $F(w) = \{x(y')\}$, as desired.

It is straightforward to check that, since ask prices for widgets such that $i \in S'$ are competitive, and prices for widgets $i \notin S'$ are prohibitively expensive, no buyer has an incentive to deviate from w to any another bid profile inducing an assignment different from x with positive probability. Similarly, since bid prices for widgets such that $i \in S'$ are competitive, and prices for widgets $i \notin S'$ are zero, no seller has an incentive to deviate from w to any another bid profile inducing an assignment different from x with positive probability. *Q.E.D.*

PROOF OF THEOREM 3.2: Suppose that $F(w) = \{x\}$ and $w = (p_1^B, ..., p_N^B, p_1^S, ..., p_M^S)$ is a Nash equilibrium. First, we claim that surplus is driven down to zero, i.e.

$$\sum_{i \in S} \sum_{j \in B} x_{ij}(p_{ij}^B - p_i^S) = 0 \text{ for all } x \in F(w).$$

If, on the contrary, for some $j \in B$ and $i \in S$, $x_{ij} = 1$ and $p_{ij}^B > p_i^S$, then buyer j has a profitable deviation to $p_{ij}'^B = p_{ij}^B - \epsilon$ for small enough $\epsilon > 0$ so that x is still preferable for the clearing house after the deviation to any $x' \neq F(w)$. Hence, $x_{ij} = 1$ implies $p_i^S = p_{ij}^B$, and every possible match makes zero or negative surplus.

Now let S' be the subset of sellers whose goods are assigned by x, let y' be the solution to $x(y') = x$, and let $p' = (p_i^S)_{i \in S'}$. We claim that (y', p') is a competitive equilibrium for the economy (S', B). To see this, note that market clearing is guaranteed by the definition of $F(w)$. Profit maximization for each seller i at the given price p_i^S is guaranteed by the fact that since w is a Nash equilibrium, $p_i^S \geq c_i$ so that selling at the price p_i^S is at least as good as not selling. Finally, we claim that each buyer $j \in B$ maximizes utility by choosing y_j' given prices p'. For suppose not; then one of the following cases must hold:

(i) $x_{ij} = 1$ for some $i \in S$ but there is $i' \in S$ such that $h_{i'j} - p_{i'}^S > h_{ij} - p_i^S$,
(ii) $x_{ij} = 0$ for all $i \in S$ but there is $i' \in S$ such that $h_{i'j} - p_{i'}^S > 0$,
(iii) $x_{ij} = 1$ for some $i \in S$ but $h_{ij} - p_i^S < 0$.

In case (i), buyer j can deviate to $p_{i'j}'^B = p_{i'}^S + \epsilon$ and $p_{i''j}'^B = 0$ for all $i'' \neq i$ for

$$0 < \epsilon < (h_{i'j} - p_{i'}^S) - (h_{ij} - p_i^S).$$

After the deviation, the clearing house should match j and i' since every other match makes zero or negative surplus by the previous step. Similarly, in case (ii), buyer j can deviate to $p_{i'j}'^B = p_{i'}^S + \epsilon$ and $p_{i''j}'^B = 0$ for all $i'' \neq i$ for

$$0 < \epsilon < h_{i'j} - p_{i'}^S.$$

As in the previous case, the clearing house should match j and i' after the deviation. In case (iii), buyer j can benefit by deviating to $p_{i'j}'^B = 0$ for all i', which guarantees a payoff of zero. Hence, in each of the three cases, w cannot be a Nash equilibrium. *Q.E.D.*

APPENDIX B: PROOFS FOR CLAIMS ABOUT RELATIONSHIPS BETWEEN CONCEPTS

This section sketches the proofs for the relationships in Figure 2.

First, $NBS \subseteq PO$ follows from homogeneity of degree 1 of the objective function in 3.3, which is therefore lower for any Pareto dominated allocation leading to a contradiction.

CLAIM B.1: $Z = NE \cap PO$.

PROOF: For the allocations without unmatched sellers and buyers, the core is $NE \cap PO$ by definition of the core and $Core \subseteq Z$ by construction of Z. For the allocations with unmatched sellers and buyers, in any Pareto optimal outcome any unmatched pair of sellers and buyers can only have 0 in the corresponding element of matrix A, otherwise the allocation is Pareto dominated by the similar one where this pair matches. Then $Z = NE \cap PO$ by construction. *Q.E.D.*

CLAIM B.2: $NBS \cap NE = Core^*$.

By Theorem 3.1 $Core \subseteq NE$. The next theorem proves that $Core^* \subseteq NBS$.

THEOREM B.1: *Every feasible efficient imputation (and thus also every core imputation) with strictly positive payoffs is an asymmetric Nash Bargaining solution (NBS) for some vector of weights $\alpha \gg 0$.*

PROOF OF THEOREM B.1: Suppose $u^* \in Core^*$, attainable at some core assignment x. Let $\hat{U}$ be the maximized surplus obtained in the core. Consider the asymmetric Nash bargaining problem of sharing $\hat{U}$ between all agents with weight vector a, $a_i = \frac{u_i^{S*}}{\hat{U}}, a_j = \frac{u_j^{B*}}{\hat{U}}$, for $i \in S$, $j \in B$, disregarding the roles and market structure. This is a standard convex optimization problem over a compact set:

$$\max_{u: \sum_{i \in S} u_i^S + \sum_{j \in B} u_j^B \leq \hat{U}} \prod_{i \in S} (u_i^S)^{\alpha_i} \prod_{j \in B} (u_j^B)^{\alpha_j}, \tag{B.1}$$

Solutions to this problem are well-defined and can be obtained from first order conditions: $u_i^S = a_i \hat{U} = u_i^*$ and $u_j^B = a_j \hat{U} = v_j^*$ for all $i \in S, j \in B$. Since $U \subseteq \{u : \sum_{i \in S} u_i^S + \sum_{j \in B} u_j^B \leq \hat{U}\}$, the imputation u^* also maximizes the Nash product over a smaller set U. Therefore u^* is the asymmetric Nash bargaining solution for weights a. *Q.E.D.*

CONTINUING PROOF FOR CLAIM B.2: We have $Core^* \subseteq NE \cap NBS$ above. For the reverse notice that the value of the objective in 3.3 is 0 if there are any unmatched players or players with 0 payoff. Thus in all imputations in NBS all players are matched. The only Nash equilibria in this case are the Core imputations by theorem 3.2. Excluding the imputations with zero payoffs, which are not in NBS, we obtain the set $Core^*$. Thus $NBS \cap NE \subset Core^*$. *Q.E.D.*

Theorem B.1 does not extend to other Pareto optima that do not maximize the total surplus. For the fact that in general $NBS \neq PO$ please see the counterexample in Appendix C.

CLAIM B.3: $Z^* = NE \cap NC$.

PROOF: If there are unmatched seller $i \in S$ and buyer $j \in B$ with $a_{ij} > 0$ then the outcome is not Pareto optimal and is also not in NC since $v^{\text{PN}}(ij) > 0$. Thus $NC \subset Core^* \cup Z^*$.

For the reverse $Z^* \subseteq NE$ by construction. It remains to show that $Z^* \subseteq NC$. Take any $u \in Z^*$ and let k be the number of unsold goods, $\underline{j} = \arg\min_{j \in B} v_j$, and $M = \sum_{k \in S \cup B, m(k) \neq \emptyset} u_k + \sum_{k \in S, m(k) = \emptyset} (A_{ij} - v_{\underline{j}}) + |k \in B, m(k) = \emptyset|$. Take weights $\theta \gg 0$ such that $\theta_i = \frac{1}{M} u_i$ and $\theta_j = \frac{1}{M} v_j$ for any matched pair $(i,j) \in B \times S$ and $\theta_{i'} = \frac{1}{M}(A_{ij} - v_{\underline{j}})$ for all non-traded goods $i' \in S$ and $\theta_{j'} = 1$ for any unmatched $j' \in B$. By choice of M the sum of weights is 1, and $\theta_i / \theta_j = u_i / v_j$ for any matched pair (i,j). Suppose $u \notin NC$. Then for some $(i,j) \in S \times B$, we have $v_i^{\text{PN}}(i,j) > u_i^S$ and $v_j^{\text{PN}}(i,j) > u_j$. But $v_i^{\text{PN}}(i,j) + v_j^{\text{PN}}(i,j) = A_{ij}$ so $u_i^S + u_j^B < A_{ij}$ and i,j can profitably match and $A_{ij} > 0$ since payoffs are nonnegative. Then since $Z^* \subseteq Z$, by definition of Z either i or j has to be matched. If i is matched then u is not a competitive equilibrium for the traded goods and is therefore not an NE by Theorem 3.1. But $Z^* \subseteq NE$ and therefore $u \notin Z^*$. If i is unmatched $v_j^{\text{PN}}(i,j) > u_j^B > 0$, but $v_j^{\text{PN}}(i,j) = \frac{\theta_j}{\theta_i + \theta_j} A_{ij} = u_{\underline{j}}^B$ and since $u_{\underline{j}}^B \leq u_j^B$ for any $j \in B$, this is a contradiction. Therefore $u \in NC$.

Q.E.D.

In simpler terms the proof states that we can explain any unsold goods by claiming that the sellers demand a very high proportion of surplus, i.e. we can assume that their bargaining weights are sufficiently high. Then, similarly to Nash equilibria, as long as the prices of traded goods are rationalizable for some weights, the whole imputation is rationalizable by the Pairwise Nash core.

APPENDIX C: NASH BARGAINING SOLUTIONS FOR THE NON-CONVEX GAME

We will illustrate the Nash bargaining extensions with the following assignment market:

$$A = \begin{bmatrix} 10 & 2 \\ 2 & 1 \end{bmatrix}$$

Consider the leximin payoff vector $u^{lm} = (u_1^S, u_2^S, u_1^B, u_2^B) : (1, 1, 1, 1)$ attainable in the circled matching — the worst-off agent cannot obtain a payoff larger than 1 in this or any other matching. The leximin solution is by definition Pareto optimal; however, it does not have to be an asymmetric Nash bargaining solution or be in the core. In this case it is not in the core since it does not maximize surplus, and it cannot be supported by any asymmetric Nash bargaining solution, which can be checked by solving the first order conditions. First, note that the value of the Nash product is 1 and that the weights of the matched buyers and sellers should be the same since the surplus is divided equally. Combining this with the normalization $\sum \theta = 1$, only one degree of freedom remains. For there to be a Nash bargaining solution, the Nash product value in the other matching should be less than 1. By solving the first order conditions for 3.3 in terms of for example θ_2, the weight of the second seller, one finds that the Nash product equals $6.325(0.5 - \theta_2)^{1-2\theta_2}\theta_2^{2\theta_2}$, the expression that is no less than 1.5 for $0 < \theta_2 < 0.5$. Thus there is no Nash bargaining weights that support u^{lm}.

The Figure C.1 best illustrates the buyers' payoffs for this market with a projection of the feasible set into a plane where the payoffs of the first seller and the first buyer are the same, and the payoffs of the second seller and the second buyer are the same. The red and green regions correspond to the feasible allocations in this plane depending on the chosen matching. The union of these regions is clearly non-convex.

The thick line illustrates part of the (strongly) Pareto optimal set under lotteries (the convex hull).

There are at least three approaches to dealing with Nash bargaining for such non-convex problems. The difference between these is contained in the non-convex regions with imputations that are Pareto dominated by lotteries such as u^{lm}. Whether these imputations are

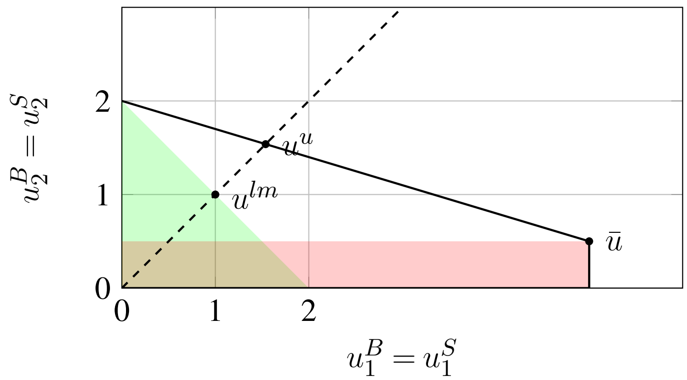


FIGURE C.1.—The three Nash bargaining solutions

reasonable outcomes depends on the chosen assumptions. The most straightforward solution is to add lotteries directly to the bargaining set, which may not be satisfactory if agents do not have expected utility preferences. Then for some weights of the asymmetric solution the point $u^u = (\frac{20}{13}, \frac{20}{13}, \frac{20}{13}, \frac{20}{13})$ that dominates u^{lm} becomes the Nash bargaining prediction even thought it suggests a probability mixture of the two matchings.

Another approach is a direct extension of the Nash Bargaining solution similar to Kaneko (1980) and Herrero (1989). That is, fall back to a set-valued solution concept that mechanically returns all maximizers to the weighted Nash function. Clearly all such points will be feasible, but there are some Pareto efficient points that cannot be supported by this concept, such as u^{lm}. With this approach some other nearby point or points would be chosen instead, e.g. with equal weights, the imputation $\bar{u} = \{5, 5, \frac{1}{2}, \frac{1}{2}\}$ is the solution. This approach is referred to as the Nash bargaining solution in this paper and corresponds to the NBS set in Figure 2.

Finally, the axiomatic approach is to follow Conley and Wilkie (1996) and consider the unique weakly Pareto optimal point on the ray (shown as a dashed line) from the disagreement point to the Nash bargaining solution for the same asymmetric bargaining problem on a convex hull of the feasible set described above (called the utopia point u^u). This set of asymmetric Nash bargaining solutions will span the whole (strongly) Pareto optimal set as one shifts the utopia point. This can be shown by extending Proposition 2 in Miyakawa (2008) using the fact that a ray from the disagreement point uniquely maps a point on the convex hull to any point on the weak Pareto frontier of the feasible set. In the same example, the weights $\{3/26, 5/13, 3/26, 5/13\}$ will lead to u^u in the problem for lotteries, which in turn is on a ray with u^{lm}.

There is a reverse problem with this approach however. In addition to he Pareto set, the set of solutions may include points that are only *weakly* Pareto optimal even when weights are strictly positive. To avoid these problems, the multi-objective optimization literature suggests simply checking every resulting solution for strong Pareto optimality and discarding the rest.

The choice between the three approaches depends on whether the outcomes like u^{lm} are deemed less likely because of the necessary sacrifice of efficiency for fairness and some choice rule or alternation between extreme but unfair outcomes like $\bar{u}$ or u_u is likely to happen instead. If yes, we need only consider NBS, if not - the whole set PO. All theories are extremely weak however spanning the larger part or the whole Pareto frontier.

The structure of the feasible and Pareto sets in the Figure C.1 is also why most of the non-core outcomes are Pareto dominated by lotteries (on the black convex hull) despite being Pareto-optimal in the actual feasible set without lotteries (the red and green regions). This suggests that subjects are not considering lotteries as a possibility.

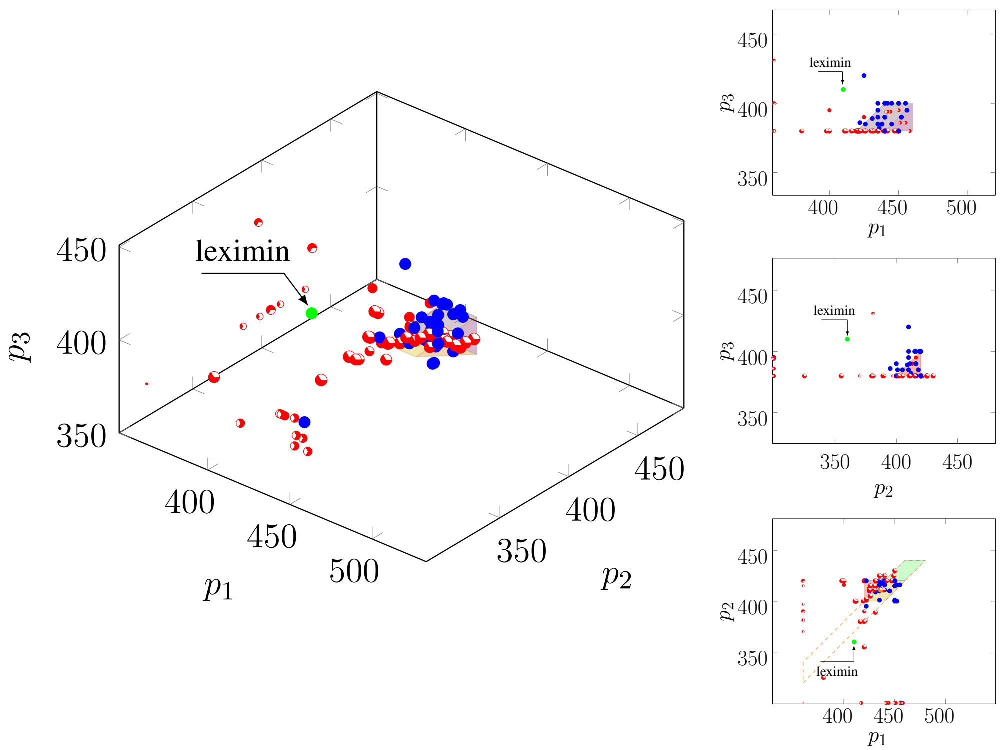
APPENDIX D: PRICES IN EXPERIMENTAL MARKETS

The four panels in Figure D.1 summarize prices for all observations in each of the treatments. The main picture in each of the panels presents the observed price vectors in three dimensions, as well as the core area, shaded in the picture. The smaller pictures present the projection of observed price vectors and the core area in two dimensions, for each pair of markets. The planes limiting the core area in the main picture, as well as the lines limiting the core area in the (projected) smaller pictures, represent various constraints on competitive prices. Intuitively, these constrains make sure that pairs of traders that should not be optimally assigned do not have a profitable opportunity to trade with each other, and that each trader prefers to trade rather than not. Slanted lines in the smaller pictures indicate are related to possible deviations from the core by pairs of traders, while lines that are parallel to either axis are related to participation constraints for the traders. Blue (entire) dots indicate complete assignments, while red dots indicate partial assignments, with the blank space indicating widgets that were not traded.

Panel (a) in Figure D.1 illustrates that observed price vectors for complete assignments under the DA treatment are either in or very close to the core area. As the smaller pictures highlight, observed price vectors are often close or above various constraints. This is clear in particular for the price of widget 3 in relation to the price of the other widgets. This seemingly indicates resistance of seller 3 to competitive forces pushing down the price of the corresponding widget. Intuitively, in the optimal assignment, when the price of widget 3 rises too much, buyer 1 would be tempted to deviate and acquire instead widget 1 or widget 2.

Panel (b) in Figure D.1 illustrates that the observed distribution of price vectors under the MP treatment is similar to that under the DA treatment, with a couple of exceptions. Constraints involving the prices of widgets 2 and 3 seem to be binding.

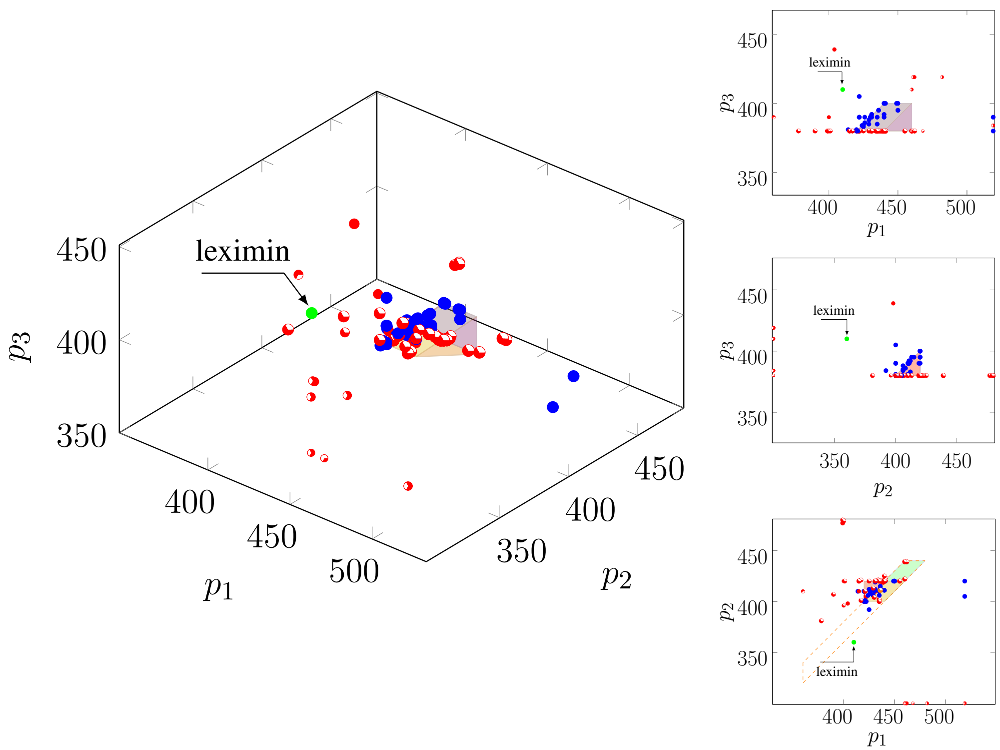
Panel (c) in Figure D.1 illustrates a distribution of price vectors under the PT treatment which differs notably from the other two treatments. There are quite a few observed price vectors for complete assignments which are not close to the core. As illustrated by the smaller pictures, observed prices tend to violate core constraints in two particular dimensions: in several observations, the price of widget 3 is too high, and the price of widget 2 is too low. This is consistent with some of the observed complete market observations corresponding to the third best assignment $[132]$. In particular, leximin solution prescribes this assignment with the price vector $(460, 360, 410)$, which violates core constraints precisely because the price of widget 3 is too high, and the price of widget 2 is too low, respect to competitive prices.



(a) Double Auction[a]

FIGURE D.1.—Resulting prices across treatments

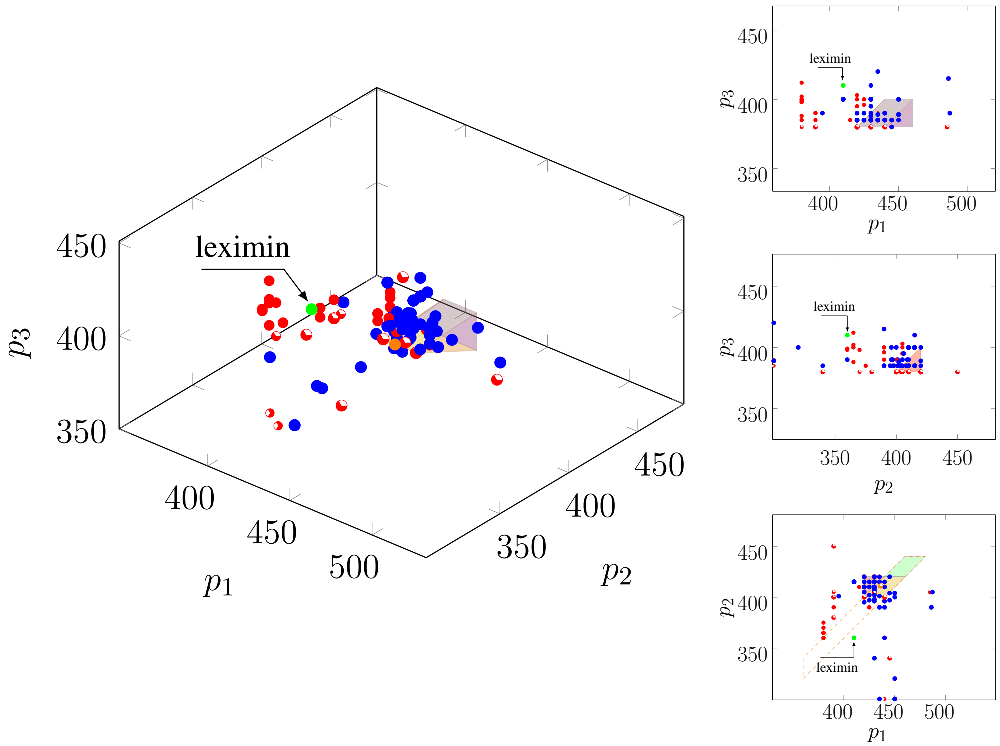
[a]Blue points are rounds with maximized total utility, filling of the markers corresponds to items traded, size of the markers is the total utility. When item is not traded, the minimum price is used instead of the trading price. The green point is the leximin price vector, the green parallelogram in the bottom panel is the core for the game without the third seller, the orange dashed region - the core for the game without the third seller and the third buyer.



(b) Minimum Price

FIGURE D.1.—Resulting prices across treatments[a]

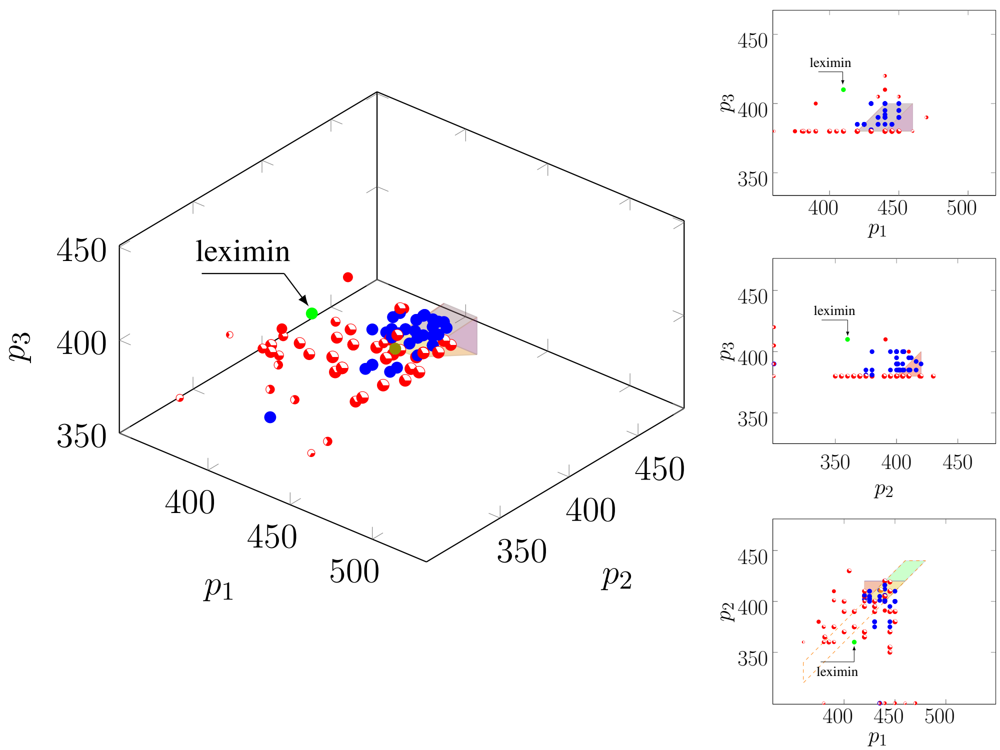
[a]Blue points are rounds with maximized total utility, filling of the markers corresponds to items traded, size of the markers is the total utility. When item is not traded, the minimum price is used instead of the trading price. The green point is the leximin price vector, the green parallelogram in the bottom panel is the core for the game without the third seller, the orange dashed region - the core for the game without the third seller and the third buyer.



(c) Pit Trading[a]

FIGURE D.1.—Resulting prices across treatments

[a]Blue points are rounds with maximized total utility, filling of the markers corresponds to items traded, size of the markers is the total utility. When item is not traded, the minimum price is used instead of the trading price. The green point is the leximin price vector, the green parallelogram in the bottom panel is the core for the game without the third seller, the orange dashed region - the core for the game without the third seller and the third buyer.



(d) Pit Trading with Complete Information

FIGURE D.1.—Resulting prices across treatments[a]

[a]Blue points are rounds with maximized total utility, filling of the markers corresponds to items traded, size of the markers is the total utility. When item is not traded, the minimum price is used instead of the trading price. The green point is the leximin price vector, the green parallelogram in the bottom panel is the core for the game without the third seller, the orange dashed region - the core for the game without the third seller and the third buyer