

MWP 2021/06  
Max Weber Programme

# WORKING PAPER

**Learning and Acyclicity in the Market Game**

Arthur Dolgoplov and César Martinelli



European University Institute  
**Max Weber Programme**

## **Learning and Acyclicity in the Market Game**

Arthur Dolgopolov and César Martinelli

MWP Working Paper 2021/06

ISSN 1830-7728

© Arthur Dolgopolov and César Martinelli, 2021

This work is licensed under a [Creative Commons Attribution 4.0 \(CC-BY 4.0\)](#) International license.

If cited or quoted, reference should be made to the full name of the author(s), editor(s), the title, the series and number, the year and the publisher.

Published in September 2021 by the European University Institute.

Badia Fiesolana, via dei Roccettini 9  
I – 50014 San Domenico di Fiesole (FI)  
Italy

[www.eui.eu](http://www.eui.eu)

Views expressed in this publication reflect the opinion of individual author(s) and not those of the European University Institute.

This publication is available in Open Access in [Cadmus](#), the EUI Research Repository:



With the support of the  
Erasmus+ Programme  
of the European Union

The European Commission supports the EUI through the European Union budget. This publication reflects the views only of the author(s), and the Commission cannot be held responsible for any use which may be made of the information contained therein.

**Abstract**

We show that strategic market games, the non-cooperative implementation of a matching with transfers or an assignment game, are weakly acyclic. This property ensures that many common learning algorithms will converge to Nash equilibria in these games, and that the allocation mechanism can therefore be decentralized. Convergence hinges on the appropriate price clearing rule and has different properties for better- and best-response dynamics. We tightly characterize the robustness of this convergence in terms of so-called schedulers for both types of dynamics.

**Keywords**

Strategic market game, Scheduler, Assignment games, Evolutionary dynamics, Weakly acyclic games.

**JEL codes:** C62, C72

*The authors would like to thank Heinrich Nax, Giacomo Calzolari, Marina Sanchez del Villar, Yann Delaprez, Konuray Mutluer, and the EUI microeconomics working group for detailed comments and questions.*

*Arthur Dolgoplov  
MWF 2020-2021 and 2021-2022  
Department of Economics, European University Institute*

*César Martinelli  
Interdisciplinary Center for Economic Science, George Mason University*



## 1. INTRODUCTION

Convergence to pure Nash equilibria is considered a fundamental problem for at least two reasons: distributed computation of equilibria, and robustness in the sense that simple agents will reach these outcomes by trial and error. Matching and assignment games (Shapley and Shubik, 1971) have been shown recently to converge for different evolutionary dynamics, but these results are embedded in a cooperative framework and are defined in terms of an explicit process of forming and breaking coalitions. We complement this analysis by putting the problem in a non-cooperative framework and show that a strategic market game (Dubey, 1982; Simon, 1984; Benassy, 1986) – in which strategic players bid competitively and publicly for a set of indivisible goods – is also weakly acyclic, converges to Nash equilibrium under mild assumptions, and can therefore be decentralized. A notable difference in approaches is that in a market game the set of equilibria is larger than the competitive outcomes of an assignment game, and players may converge to inferior semi-Walrasian outcomes of Mas-Colell (1982).

Convergence property depends on the choice of dynamics, the process according to which agents update their strategies. A natural dynamic to consider for a non-cooperative game is a better- (best-) response dynamic. Under this process players sequentially (one by one, in random order) shift to one of the better (best) responses to the current action profile. A sufficient condition for convergence of better-response dynamics is *weak acyclicity*: for any profile there is a path of better responses leading to a pure Nash equilibrium.

Many important classes of games exhibit weak acyclicity including all generalized ordinal potential games (Monderer and Shapley, 1996). The closest results to this paper, however, lie in the cooperative game theory literature that directly studies matching problems. Convergence has been shown for models of two-sided matchings

with non-transferable utility (Ackermann et al., 2011), and moreover a subset of stochastically stable outcomes can be easily identified (Newton and Sawa, 2015). Convergence for transferable utility matchings and assignment games under different plausible dynamics has also been shown in multiple studies (Klaus et al., 2010; Chen et al., 2010; Nax et al., 2013; Klaus and Newton, 2016) and refined to stochastic stability under perturbed dynamics (Klaus and Newton, 2016; Nax and Pradelski, 2015). Nax (2019) shows the results for uncoupled dynamics where players have no information about other players' payoffs. The analysis in (Nax and Pradelski, 2015) is particularly close to the present paper and our proofs largely follow the same path with two important differences. Since we refine weak acyclicity and state the results in terms of schedulers (discussed below) we have to construct the path to the equilibrium explicitly, while the weak acyclicity considered by Nax and Pradelski (2015) only requires showing that some path exists. The second difference is that, due to our non-cooperative framework and different rematching process, the latter does not have to converge to the core (please see Appendix B for the exact characterization of equilibria).

The convergent process can also be seen as similar to the mechanism in Demange et al. (1986) but there is a subtle difference. To show convergence, we need to show acyclicity not only on path or in the main phase of the mechanism in Demange et al. (1986), but also everywhere off the equilibrium path, including the profiles where buyers are bidding too much and for possibly suboptimal items. In particular we must consider the cases where the highest bidding buyers switch goods, even though this would never happen if the game started at zero prices and proceeded according to the best-response dynamics. This is because we want to guarantee convergence of perturbed learning algorithms that explore the strategy space and learn through experimentation, and thus reach any outcome within the strategy space with positive probability. In fact the game is likely to start at a Nash equilibrium; no-trade is

(usually) also a Nash equilibrium. It is crucial therefore to explore convergence of algorithms that involve some experimentation, so that they would be able to move between the sinks of best responses that are Nash equilibria, and would be able to start trading.

It is possible to refine weak acyclicity for both better and best responses. [Apt and Simon \(2015\)](#) provide a classification of refinements of weak acyclicity up to finite improvement property (i.e. potential games), introducing different *schedulers* for the classes in between. A scheduler is a function that somehow selects a player who can shift to a better response in each profile to get out of best- or better-response cycle. The complexity of this function (and the respective class of schedulers) indicates robustness of the dynamic process. For example, a local scheduler only requires that an agent is selected at each profile according to some fixed arbitrary order (e.g. bargaining power of agents). A state scheduler is more general, and can select the deviating player based on the information from the state of the game (action profile), e.g. the player with the highest payoff etc. Therefore, if a game requires a state scheduler and does not admit a local scheduler, convergence in this game is, in this particular sense, less robust.

The concept of a scheduler is also attractive because it indicates how difficult it is for the agents to choose the improvement path that leads out of cycles and towards the equilibrium. A set-based scheduler would indicate that this process only requires that agents act in a particular order (for a local scheduler) or, at least that the deviating agent is chosen independently of the action profile (for a general set scheduler).

We place both better and best responses for market games within the classification of [Apt and Simon \(2015\)](#). For better responses we show that only the most general, state scheduler exists. For best-response dynamics we can claim a stronger result; that a market game admits *any* local scheduler. In other words, as long as the order

of deviations is predetermined (e.g. implicitly by bargaining power of the players or in any arbitrary way) convergence is guaranteed.

The paper is organized as follows. The next section introduces the definitions of the market game, the schedulers and related concepts. The third section contains the main results for convergence and counter-examples and is split into two parts for best-response and better-response dynamics. Finally the last fourth section discusses an alternative market clearing rule and concludes. Since proofs are constructive and notation-heavy, they are collected in Appendix A. Appendix B characterizes Nash equilibria, the sinks of the convergent dynamics, in terms of competitive outcomes.

## 2. PRELIMINARIES

**2.1. Strategic market games.** Let  $B$  and  $S$  denote respectively the sets of buyers and sellers, with  $|B| = N$  and  $|S| = M$ . Each seller has one good to sell, and therefore we use  $S$  for the set of goods as well. Each buyer  $j$  has a valuation for each seller's good  $i$ , denoted by  $h_{ij}$ , while each seller has a reservation value for his own good  $c_i > 0$ . We refer to  $\{h, c\}$  as an *economy*. We will call a smaller economy  $\{h', c'\}$  restricted to goods  $S' \subset S$  in a natural way with  $h_{ij} = h'_{ij}$  and  $c'_i = c_i$  for all  $i \in S'$  and  $j \in B$  a *subeconomy for goods  $S'$* .

We will use  $i$  for a representative seller,  $j$  for a buyer, and  $k$  for some player, either a buyer or a seller. In the strategic *market game* each seller  $i \in S$  submits a price  $p_i^S \in \mathbb{R}_+$ , and each buyer submits an index of a seller  $m_j \in S$  and a positive bid for her good. We will write the buyer's action as an  $M$ -vector with at most one strictly positive element,  $p_j^B \in \mathbb{R}_+^M, p_j^B = (p_{1j}^B, \dots, p_{Mj}^B)$ . We also implicitly define  $m_j(p)$  as a function that gives the good that the buyer  $j$  is bidding for in profile  $p$ . Note that she does not necessarily obtain this good in  $p$ .

The set of admissible prices/bids for player  $k \in B \cup S$  is denoted  $\Omega_k$ . These sets are finite with bids and prices chosen on a grid with  $\epsilon$  between consecutive values.

We will assume without loss of generality that  $\epsilon = 1$  and bids and prices therefore have to be integers,  $\Omega_k \subset \mathbb{Z}$  for all  $k$ . Likewise all  $h_{ij} \in \mathbb{Z}$  for all  $i \in S$  and  $j \in B$ .

An *action profile* combines actions of all players  $p = (p_1^B, \dots, p_N^B, p_1^S, \dots, p_M^S) \in \Omega = \prod_{k \in B \cup S} \Omega_k$ . The action of a generic player  $k$  is just  $p_k$ . For simplicity dominated actions – bids above valuations and seller prices below costs – are not allowed and are excluded from  $\Omega_k$  and  $\Omega$ .

Once all bids and prices are submitted, a clearing house chooses an assignment in the feasible set

$$X = \{(x_{11}, \dots, x_{NM}) : x_{ij} \in \{0, 1\}, \sum_{j \in B} x_{ij} \leq 1 \text{ for all } i \in S\}.$$

The clearing house allocates trade to maximize surplus

$$\Xi(x, p) = \sum_{i \in S} \sum_{j \in B} x_{ij} (p_{ij}^B - p_i^S).$$

In other words, it draws from the following set of surplus-maximizing assignments:

$$\bar{\Pi}(p) = \{x \in X : \Xi(x, p) \geq \Xi(x', p) \text{ for all } x' \in X\}.$$

To ensure that the clearing house prefers more trade even when arbitrage is zero, we assume that it chooses assignments that are not ray-dominated ([Simon, 1984](#)):

$$\Pi(p) = \{x \in \bar{\Pi}(p) : \text{there is no } \hat{x} \in \bar{\Pi}(p) \text{ such that}$$

$$\hat{x} \neq x \text{ and } \hat{x}_{ij} \geq x_{ij} \text{ for all } i \in S, j \in B\}.$$

Once the clearing house chooses an assignment  $x$ , the market clears at the respective prices of buyers and sellers. That is, the payoffs are defined by the following rule that

we call **s-prices**:

$$\text{(s-prices)} \quad u_j^B(p, x) = \max_{i \in S} (x_{ij} h_{ij}) - \sum_{i \in S} p_{ij}^B x_{ij} \quad \text{and} \quad u_i^S(p, x) = \sum_{j \in B} (p_i^S - c_i) x_{ij}.$$

We will discuss the alternative market clearing rule (and why it does not work) in the last section.

**2.2. Dynamic components.** The following definitions are taken from [Apt and Simon \(2015\)](#). An action  $p'_k$  of player  $k$  is a *better response* from an action profile  $p$  if  $u_k(p'_k, p_{-k}) > u_k(p_k, p_{-k})$  and a *best response* from an action profile  $p$  if  $u_k(p'_k, p_{-k}) \geq u_k(p_k, p_{-k})$  for all  $p'_k \in \Omega_k$ . A *path* in  $\Omega$  is a sequence  $(p^1, p^2, \dots)$  of action profiles such that for every  $l > 1$  there is a player  $k$  such that  $p^k = (p'_k, p_{-l}^{l-1})$  for some  $p'_k \neq p_k^{l-1}$ . Player  $k$  is then said to have *deviated* from  $p^{l-1}$ . A path is called an *improvement path* (a best response improvement path, shortened to BR-improvement path) if for all  $l > 1$ ,  $p_k^l$  is a better (best) response to  $p_{-k}^{l-1}$ , where  $k$  is the player who deviated from  $p^{l-1}$ . The sets of improvement paths and BR-improvement paths from a profile  $p$  are written as  $\mathcal{P}(p)$  and  $\mathcal{P}^{BR}(p)$ .

Following [Young \(1993\)](#); [Milchtaich \(1996\)](#), and [Apt and Simon \(2015\)](#) we say that the game has the finite improvement property or FIP (respectively, the finite best response property, FBRP) if every improvement path (BR-improvement path) is finite. The class of games with FIP is exactly the class of generalized ordinal potential games ([Monderer and Shapley, 1996](#)). A strategic market game is *weakly acyclic* (respectively, BR-weakly acyclic) if for any action profile there exists a finite improvement path (BR-improvement path) that starts at it.

A *scheduler* (denoted  $f$ ) is a function that given a finite sequence of profiles  $p^1, \dots, p^k$  that does not end in a Nash equilibrium selects a player who did not select a best response in  $p^k$ . That is, the scheduler defines the rule of picking the next deviating

player at each profile with the goal of avoiding cycles. A BR-scheduler is a scheduler applied to BR-improvement paths. We will say  $f(p) = \emptyset$  if  $p$  is a Nash equilibrium.

We say that an improvement path  $\rho = (p^1, p^2 \dots)$  *respects* a scheduler  $f$  if for all  $l < |\rho|$  we have  $p^{l+1} = (p'_k, p^l_{-k})$  where  $f(p^1, \dots, p^l) = k$ . We say that a strategic game *respects a scheduler*  $f$  if all improvement paths  $\rho$  that respect  $f$  are finite, and similarly for a BR-scheduler.

We now define different kinds of schedulers from less to more restrictive.

A scheduler  $f$  is *state-based* if for some function  $g : \Omega \rightarrow N$  we have

$$f(p^1, \dots, p^l) = g(p^k).$$

This is the most general class of schedulers, i.e. all schedulers are state-based.

The function  $g : 2^{(S \cup B)} \rightarrow (S \cup B)$  is a *choice function* if for all  $A \neq \emptyset$  we have  $g(A) \in A$ . A scheduler  $f$  is *set-based* if for some choice function  $g$ :

$$f(p^1, \dots, p^l) = g(NBR(p^l)).$$

Finally, a set-based scheduler  $f$  is *local* if  $g$  also satisfies

$$g(A) \in B \subseteq A \implies g(A) = g(B).$$

A local scheduler  $f$  can be equivalently defined in terms of some strict total order  $\prec_f$ :

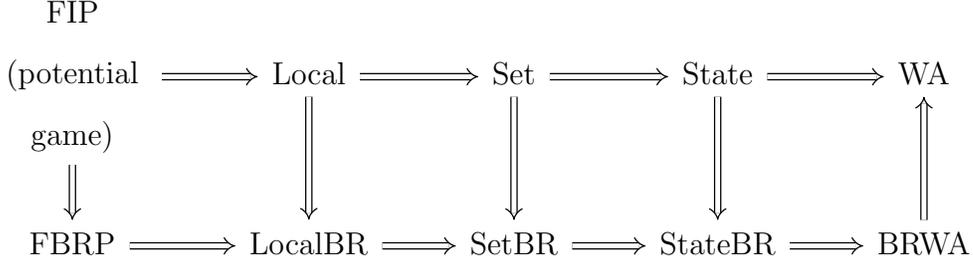
$$f(p) = k \in \{k \prec_f k', \forall k' \in I(p)\}.$$

That is, deviating players are chosen according to some predefined priority rule<sup>1</sup>. Since  $\prec_f$  is a strict total order, such  $k$  is unique. We will also write  $j \preceq_f k$  for “ $j \prec_f k$  or  $j = k$ ”.

---

<sup>1</sup>Using  $\prec_f$  for a local scheduler is without loss of generality by Proposition 1 in [Apt and Simon \(2015\)](#), where this order is captured by permutation  $\pi$ .

Apt and Simon (2015) provide a classification of refinements of weak acyclicity up to finite improvement property (i.e. potential games) using the schedulers defined above with inclusions as follows:



It will be convenient to talk about schedulers in terms of potentials. A function  $F$  is an  $f$ -potential iff for all  $k \in B \cup S$ , and  $p'_i, p \in \Omega$ :

if  $f(p) = k$  and  $u_k(p'_k, p_{-k}, \pi(p'_k, p_{-k})) > u_i(p_i, p_{-i}, \pi(p_i, p_{-i}))$ , then

$$(p'_i, p_{-i}) < F(p_i, p_{-i}).$$

A scheduler can be thought of as giving “priority” to some set of players, either sellers or current top bidders etc. Denote the set of players who are not playing a best response in profile  $p$  by  $I(p)$ . We will say that a scheduler (BR-scheduler) *prioritizes* some set of players  $\uparrow_f(p)$  over another set of players  $\downarrow(p)$  if  $f(p) \notin \downarrow(p)$  whenever  $I(p) \cap \uparrow(p) \neq \emptyset$ . For a local scheduler  $f$  with an associated strict total order  $\prec_f$  this implies that  $k \prec_f k'$  for any  $k \in \uparrow(p)$  and  $k' \in \downarrow(p)$ , and for all  $\Omega$ . For example, with  $\uparrow(p) = S$  and  $\downarrow(p) = B$  and  $i \prec_f j$  for any  $i \in S$  and  $j \in B$  the scheduler  $f$  prioritizes sellers over buyers. Notice that  $\uparrow$  and  $\downarrow$  can also be functions of the profile, e.g. the set of top bidders. We will thus, with abuse of notation, write  $\uparrow(p) \prec_f \downarrow(p)$  meaning that given a profile  $p$ ,  $f$  prioritizes  $\uparrow(p)$  over another set of players  $\downarrow(p)$  even if  $f$  is not local.

**2.3. Tie-breaking assumptions.** Nash equilibria of a market game can include ties, and thus  $h$  and  $c$  are not by themselves enough to define a market game. We assume

that the clearing house chooses the buyer in any tie in  $\Pi(p)$  by randomizing over the full support. Only the buyers who obtain the good in the realized outcome pay its price. Every seller  $i$  has a weight for buyer  $j$  denoted by  $\pi_{ij}$ . The probability of obtaining a good  $i$  by any buyer  $j$  at her winning bid  $p_{ij}^B$  is proportional to the seller's weight for this buyer. In other words it equals

$$\frac{\pi_{ij}}{\sum_{j' \in B, p_{ij'}^B = p_{ij}^B} \pi_{ij'}}.$$

The probability of any outcome  $x \in \Pi(p)$ , denoted by  $Pr(x|p)$ , is then the product of these expressions. Note that the clearing house is free to break ties differently for each seller, or, in other words, sellers can break ties themselves according to their own unique vectors of weights. We will denote the expected utility of player  $k$  in profile  $p$  by  $U_k(p) = U_k(p_k, p_{-k}) = \sum_{x \in \Pi(p)} Pr(x|p)u(p, x)$ , where  $p_{-k}$  are offers of all sellers and all other buyers  $B \setminus k$ .

We assume a small level of risk-aversion to make sure that two players with equal valuations  $h_i$  for the same good  $i$  would not tie for it at a price  $h_i - 2$ , i.e. two steps away from the competitive price. The following assumption says that in this case, each of them prefers to take the good for herself definitively by bidding 1 more, even though the payoff is the same in expectation.

**Assumption 1.** *Buyers exhibit risk-aversion when they compare two outcomes of equal expected utility. In other words, any buyer  $j$  prefer  $p$  to  $p'$  if either  $U_j(p) > U_j(p')$  or  $U_j(p) = U_j(p')$  and  $|\Pi(p)| < |\Pi(p')|$ .*

Although tie-breaking issues can generally expand the set of Nash equilibria of a market game, they can be contained quite well. It only affects the prices in the negligible manner, and does not to affect the matchings at all as long as we make sure that the valuations are no more coarse than the possible bids:

**Assumption 2** (Coarsness). *The action space is at least two times less coarse than the valuation space, i.e. for any  $i \in S$  and  $j \in B$ ,  $h_{ij} = 2z$ , for some  $z \in \mathbb{Z}$ .*<sup>2</sup>

The assumption says that the currency medium is smaller than the minimal amount required to represent the smallest relevant payoff difference and allows one to compare some tied outcomes as well.

### 3. CONVERGENCE RESULTS

**3.1. Best responses.** We first show that even when agents are restricted to best responses, the market game is not generally a potential game. In all examples assume the ties to be broken uniformly.

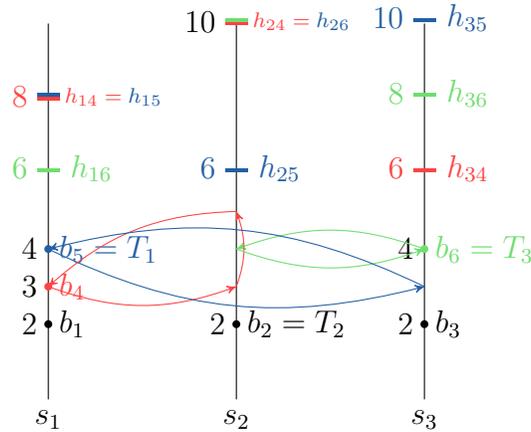
We cannot hope to achieve convergence unless we force sellers to adjust their actions before the buyers for the same reasons that we have defined **s-prices** to be the market clearing rule (see the discussion section for the counterexample). We will show a stronger counterexample here; even if we restrict sellers to always move before buyers when they have a better response, the game does not necessarily converge. An example of such a game that does not have a BR-potential, i.e. does not satisfy FBRP, is below.

**Example 1.** *Market game that has no FBRP.*

---

<sup>2</sup>To see why this assumption is necessary please consult Appendix B, which contains two theorems characterizing equilibria with and without the assumption.

$\begin{bmatrix} 2 & 2 & 2 & 8 & 10 & 6 \\ 2 & 2 & 2 & 8 & 6 & 10 \\ 2 & 2 & 2 & 6 & 10 & 8 \end{bmatrix}$									$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$												
<i>valuations h</i>									<i>costs c</i>												
$b_4$			$b_5$			$b_6$			$s_1$	$s_2$	$s_3$				$b_4$	$b_5$	$b_6$	$s_1$	$s_2$	$s_3$	
$b_4^1$	$b_4^2$	$b_4^3$	$b_5^1$	$b_5^2$	$b_5^3$	$b_6^1$	$b_6^2$	$b_6^3$					$b_4$	$b_5$	$b_6$	$s_1$	$s_2$	$s_3$			
3	0	0	4	0	0	0	0	4	4	2	4		⓪	6	⓪	3	1	3			
0	3	0	4	0	0	0	0	4	4	2	4		5	6	⓪	3	⓪	3	⓪	1	3
0	3	0	4	0	0	0	0	4	4	3	4		5	6	⓪	3	2	3			
0	3	0	4	0	0	0	4	0	4	3	4		⓪	⓪	6	3	⓪	3	⓪	2	0
0	3	0	4	0	0	0	4	0	4	4	4		⓪	⓪	6	3	3	3	⓪	3	⓪
0	3	0	4	0	0	0	4	0	4	4	2		⓪	⓪	6	3	3	3	3	1	1
0	5	0	4	0	0	0	4	0	4	4	2		3	⓪	⓪	3	⓪	3	⓪	3	1
0	5	0	4	0	0	0	4	0	4	5	2		3	⓪	⓪	3	4	4	1	1	1
0	5	0	0	0	3	0	4	0	4	5	2		⓪	7	⓪	⓪	4	1	4	1	⓪
0	5	0	0	0	3	0	4	0	2	5	2		⓪	7	⓪	1	4	4	⓪	1	⓪
0	5	0	0	0	3	0	4	0	2	5	3		⓪	7	⓪	1	4	2	⓪	2	⓪
0	5	0	0	0	3	0	0	4	2	5	3		⓪	⓪	4	1	4	4	⓪	3	⓪
3	0	0	0	0	3	0	0	4	2	5	4		5	⓪	⓪	⓪	1	0	0	3	3
3	0	0	0	0	3	0	0	4	3	5	4		5	⓪	⓪	2	⓪	0	⓪	3	3
3	0	0	0	0	3	0	0	4	3	2	4		5	⓪	⓪	2	1	3			
<i>action profiles p</i>									<i>payoffs u</i>												



The diagram and the table above show a BR-improvement path that cycles. The diagram shows the valuations (bars) and bids (dots) of all buyers for each good individually, starting with a profile where buyer 5 buys the first good, buyer 2 buys the second, and buyer 6 buys the third good. After every shift by a buyer the sellers

adjust their prices immediately and are not shown. The order of moves and the detailed BR-improvement paths can be seen in the table with circled payoffs indicating players that have a better response deviation. We use the first three buyers  $b_1, b_2, b_3$  as a gadget to make sellers drop their prices to 2 (for a positive payoff of 1) when the three actively moving buyers  $b_4, b_5, b_6$  do not bid for a particular good. The arrows connect the shifts of every buyer and can be seen to cycle. Thus, there cannot be a generalized potential for this game even when agents are restricted to best-response behavior. Moreover, in the depicted cycle sellers always move before buyers.

Nonetheless, the market game is weakly acyclic and we will prove a stronger fact that the game admits a local BR scheduler. This is the exact position of the market game within the classification of schedulers, and it implies that the game is weakly acyclic through the relationships in [Apt and Simon \(2015\)](#). In fact the game admits *any* local BR-scheduler as long as sellers move before buyers:

**Proposition 1.** *Any strategic market game admits a local BR-scheduler. Moreover, it admits any local BR-scheduler that prioritizes sellers over buyers.*

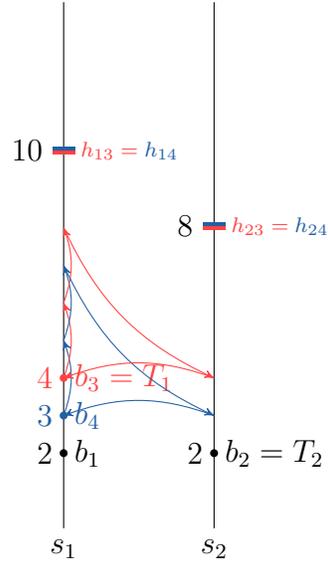
The proof of this proposition as well as the other results are collected in [Appendix A](#).

**3.2. Better responses.** In terms of better-response dynamics, some paths also cycle, i.e. a strategic market game is not a (generalized ordinal) potential game. This is implied by a stronger fact that it does not admit any local or set scheduler, in contrast with its best-response counterpart. The exact place of the better-response dynamics in the classification is the fact that the game admits a state scheduler. In other words, fixing an arbitrary order of moves is no longer enough when players do not always play best responses. We again first show a counterexample for the existence of a set scheduler and then prove the positive result for a state scheduler.

The cycle of action profiles illustrated below respects the following local scheduler:  $s_1 \prec_f s_2 \prec_f b_4 \prec_f b_3 \prec_f b_1 \prec_f b_2$  and any local scheduler obtained from it by changing the positions of buyers 1 and 2 since they do not have improvements and play their unique best responses.

**Example 2.** Market game that does not respect a local scheduler, cycle for  $b_4 \prec_f b_3$ .

$\begin{bmatrix} 2 & 2 & 10 & 10 \\ 2 & 2 & 8 & 8 \end{bmatrix}$				$\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$			
valuations $h$				costs $c$			
$b_3$	$b_4$	$s_1$	$s_2$	$b_3$	$b_4$	$s_1$	$s_2$
$b_3^1$	$b_3^2$	$b_4^1$	$b_4^2$	$b_3$	$b_4$	$s_1$	$s_2$
4	0	3	0	4	2	6	0
4	0	5	0	4	2	0	5
4	0	5	0	5	2	0	5
6	0	5	0	5	2	4	0
6	0	5	0	6	2	4	0
6	0	7	0	6	2	0	3
6	0	7	0	7	2	0	3
8	0	7	0	7	2	2	0
8	0	7	0	8	2	2	0
8	0	0	3	8	2	2	5
8	0	0	3	8	3	2	5
0	4	0	3	8	3	2	5
0	4	0	3	2	3	4	0
0	4	0	3	2	4	4	0
0	4	3	0	2	4	4	7
0	4	3	0	3	4	4	7
4	0	3	0	3	4	6	0
4	0	3	0	4	4	6	0
action profiles $p$				payoffs $u$			



Sellers' payoffs are independent of the other sellers and a similar cycle exists if we were to have  $s_2 \prec_f s_1$ . The game is also symmetric for buyers and there is another cycle for a scheduler with  $b_3 \prec_f b_4$ , which can be obtained by switching the actions of buyer 3 and buyer 4. Since there are only two buyers who can have better responses, every set scheduler that prioritizes sellers is also a local scheduler. Therefore, this game is also an example of a market game that does not admit a set scheduler.

There is however a natural way to ensure convergence for better responses by using a state scheduler.

**Proposition 2.** *Any strategic market game admits a (state) scheduler.*

The idea behind the proof of this proposition is that the scheduler ensures that the players who can potentially cause a price drop move first. These players are the top bidders that are not tied and are willing to switch to another good. If there are no such players, other top bidders  $T(p)$  move, and then the rest of the buyers who do not have a highest bid for any good. We thus separate the two phases of the convergence process, the main phase similar to (Demange et al., 1986) and the rematching of top bidders. This is the same approach as in (Nax and Pradelski, 2015) for aspiration dynamics, but with more details required by the non-cooperative setup.

#### 4. DISCUSSION

In this paper we have shown the restrictions necessary to ensure convergence of a decentralized market for indivisible goods to an equilibrium. These results can be used to refine the sets of predictions of some perturbed learning dynamic using the standard approaches to stochastic stability stemming from Young (1993); Foster and Young (1990), or to make sure that market institutions are designed with these restrictions in mind to promote the desired outcomes. We conclude by discussing the robustness of these results to several other specifications.

**4.1. Restricted action space.** Schedulers restrict the possible deviations on the route to the Nash equilibria by narrowing the set of possible deviators. A natural alternative to this might be to restrict the set of actions instead. By construction, such an approach would have to depend on a particular game and would not be as universal as schedulers. However, for the strategic market game in particular, this would not get us far. It appears reasonable to allow players to at least make a best

response (and possibly some other actions) at each profile. However, by Example 1 we know that the game does not satisfy FBRP, and therefore even if there were no other actions except for best responses, the game would not always reach an equilibrium. If we could choose to disallow best responses in some profiles instead, we could simply implement one of the optimal mechanisms like (Demange et al., 1986). Therefore, at least for the strategic market game, the possibilities for refining weak acyclicity appear to lie with the schedulers, not with restricting actions.

**4.2. Discrete bids.** We consider a discretized game with an arbitrarily large but finite set of admissible action profiles. One reason for this is that a truly continuous game would require refining the concept of better- (best-)response dynamics to avoid infinitely small price adjustment sequences. As an example, Barron et al. (2010) and Hofbauer and Sorin (2006) use differential inclusions for this purpose. However, a continuous setting is less relevant to issues in this paper and to laboratory or real-life scenarios where players are ultimately limited to finite increments of bids (at least up to machine precision). At the same time, the necessity of Assumption 2 and the fact that prices can only be one  $\epsilon$  away from competitive can be useful if  $\epsilon$  is taken to represent the coarseness of players' perception of bids and valuations. If players cannot make sufficiently precise bids, perhaps, for behavioral reasons that limit their perception of small price changes, then the adverse effects of this behavior would be limited by the Nash equilibrium characterization results in Appendix B.

**4.3. Alternative market clearing rule.** We could alternatively always clear the market at buyers' prices, i.e. replace the market clearing rule that we called **s-prices** with the following:

$$\text{(b-prices)} \quad u_j^B(p, x) = \max_{i \in S} (x_{ij} h_{ij}) - \sum_{i \in S} p_{ij}^B x_{ij} \quad \text{and} \quad u_i^S(p, x) = \sum_{j \in B} (p_{ij}^B - c_i) x_{ij}.$$

There are several problems with clearing the market at buyers' prices. In a discrete action version of the market game that clears at the highest bid the surplus no longer has to be zero in a Nash equilibrium; a simple example is any market where two buyers are willing to tie for a good at a price strictly higher than the seller's price. This is not the problem in itself, and we could focus on whether the highest bid is competitive and ignore the sellers' side altogether. However, even then, the market does not converge to the Nash equilibria as the game is no longer weakly acyclic. In other words, an active seller is necessary to guarantee convergence. The game below is an example of such a market where the cycle is the consequence of an inappropriate market clearing rule that removes incentives from the sellers.

**Example 3.** *Market game that is not weakly acyclic under  $b$ -prices.*

$\begin{bmatrix} 2 & 0 \\ 4 & 4 \end{bmatrix}$	<table style="border-collapse: collapse; margin: 0 auto;"> <tr> <td style="padding: 2px 5px;"><math>b_1</math></td> <td style="padding: 2px 5px;"><math>b_2</math></td> <td style="border-left: 1px solid black; padding: 2px 5px;"><math>s_1</math></td> <td style="padding: 2px 5px;"><math>s_2</math></td> </tr> <tr> <td style="padding: 2px 5px;"><math>b_1^1</math></td> <td style="padding: 2px 5px;"><math>b_1^2</math></td> <td style="border-left: 1px solid black; padding: 2px 5px;"><math>b_2^1</math></td> <td style="padding: 2px 5px;"><math>b_2^2</math></td> </tr> <tr> <td style="padding: 2px 5px;">1</td> <td style="padding: 2px 5px;">0</td> <td style="border-left: 1px solid black; padding: 2px 5px;">0</td> <td style="padding: 2px 5px;">1</td> </tr> <tr> <td style="padding: 2px 5px;">0</td> <td style="padding: 2px 5px;">0</td> <td style="border-left: 1px solid black; padding: 2px 5px;">0</td> <td style="padding: 2px 5px;">1</td> </tr> <tr> <td style="padding: 2px 5px;">1</td> <td style="padding: 2px 5px;">0</td> <td style="border-left: 1px solid black; padding: 2px 5px;">0</td> <td style="padding: 2px 5px;">0</td> </tr> <tr> <td style="padding: 2px 5px;">0</td> <td style="padding: 2px 5px;">1</td> <td style="border-left: 1px solid black; padding: 2px 5px;">0</td> <td style="padding: 2px 5px;">0</td> </tr> <tr> <td style="padding: 2px 5px;">0</td> <td style="padding: 2px 5px;">0</td> <td style="border-left: 1px solid black; padding: 2px 5px;">0</td> <td style="padding: 2px 5px;">0</td> </tr> <tr> <td style="padding: 2px 5px;">0</td> <td style="padding: 2px 5px;">1</td> <td style="border-left: 1px solid black; padding: 2px 5px;">0</td> <td style="padding: 2px 5px;">1</td> </tr> </table>	$b_1$	$b_2$	$s_1$	$s_2$	$b_1^1$	$b_1^2$	$b_2^1$	$b_2^2$	1	0	0	1	0	0	0	1	1	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1	<table style="border-collapse: collapse; margin: 0 auto;"> <tr> <td style="padding: 2px 5px;"><math>b_1</math></td> <td style="padding: 2px 5px;"><math>b_2</math></td> <td style="border-left: 1px solid black; padding: 2px 5px;"><math>s_1</math></td> <td style="padding: 2px 5px;"><math>s_2</math></td> </tr> <tr> <td style="padding: 2px 5px;"><math>\textcircled{1}</math></td> <td style="padding: 2px 5px;">3</td> <td style="border-left: 1px solid black; padding: 2px 5px;">0</td> <td style="padding: 2px 5px;">0</td> </tr> <tr> <td style="padding: 2px 5px;"><math>\textcircled{0}</math></td> <td style="padding: 2px 5px;">3</td> <td style="border-left: 1px solid black; padding: 2px 5px;">0</td> <td style="padding: 2px 5px;">0</td> </tr> <tr> <td style="padding: 2px 5px;"><math>\textcircled{1}</math></td> <td style="padding: 2px 5px;"><math>\textcircled{0}</math></td> <td style="border-left: 1px solid black; padding: 2px 5px;">0</td> <td style="padding: 2px 5px;">0</td> </tr> <tr> <td style="padding: 2px 5px;">3</td> <td style="padding: 2px 5px;"><math>\textcircled{0}</math></td> <td style="border-left: 1px solid black; padding: 2px 5px;">0</td> <td style="padding: 2px 5px;">0</td> </tr> <tr> <td style="padding: 2px 5px;"><math>\textcircled{0}</math></td> <td style="padding: 2px 5px;"><math>\textcircled{0}</math></td> <td style="border-left: 1px solid black; padding: 2px 5px;">0</td> <td style="padding: 2px 5px;">0</td> </tr> <tr> <td style="padding: 2px 5px;"><math>\textcircled{&lt;3}</math></td> <td style="padding: 2px 5px;"><math>&lt;3</math></td> <td style="border-left: 1px solid black; padding: 2px 5px;">0</td> <td style="padding: 2px 5px;">0</td> </tr> </table>	$b_1$	$b_2$	$s_1$	$s_2$	$\textcircled{1}$	3	0	0	$\textcircled{0}$	3	0	0	$\textcircled{1}$	$\textcircled{0}$	0	0	3	$\textcircled{0}$	0	0	$\textcircled{0}$	$\textcircled{0}$	0	0	$\textcircled{<3}$	$<3$	0	0
$b_1$	$b_2$	$s_1$	$s_2$																																																											
$b_1^1$	$b_1^2$	$b_2^1$	$b_2^2$																																																											
1	0	0	1																																																											
0	0	0	1																																																											
1	0	0	0																																																											
0	1	0	0																																																											
0	0	0	0																																																											
0	1	0	1																																																											
$b_1$	$b_2$	$s_1$	$s_2$																																																											
$\textcircled{1}$	3	0	0																																																											
$\textcircled{0}$	3	0	0																																																											
$\textcircled{1}$	$\textcircled{0}$	0	0																																																											
3	$\textcircled{0}$	0	0																																																											
$\textcircled{0}$	$\textcircled{0}$	0	0																																																											
$\textcircled{<3}$	$<3$	0	0																																																											
<i>surplus matrix</i> $A$	<i>0-arbitrage action profiles</i> $p$	<i>payoffs</i> $U$																																																												

We will show how the example works. First note that in any Nash equilibrium there is no arbitrage (by remark in the beginning of the proof of Theorem 2). At the same time sellers do not have any better responses in any profile. Fix  $p_1^S$  and  $p_2^S$  at 1. Then there are 6 possible zero-arbitrage profiles, but none of them are Nash equilibria; these are shown in the table above with circled payoffs indicating players with a deviation. In the last profile the buyers tie for good 2, but by Assumption 1, there is at least some positive probability that buyer 1 does not obtain the good. Then shifting to good 1 is an improvement for her. Since there are no better responses leading sellers to increase their prices and no equilibria at current seller prices, every chain of better responses has to eventually cycle.

## APPENDIX A. PROOFS

**A.1. Preliminaries.** Before proving Propositions 1 and 2 we will introduce several helpful lemmas and definitions, most importantly the individual “components of potentials” of players  $F_k(p)$ ,  $F_k^{BR}(p)$  that we will use in constructing the  $f$ -potential function for better and best responses respectively.

In order to constructively prove acyclicity of improvement paths it is convenient to rely on auxiliary variables of a particular form. For example, the sellers simply match the highest bid whenever they can, so it will be useful to introduce functions to describe how prices change with the sellers’ behavior. More generally, we will introduce variables that do not increase/decrease as a subset of players make deviations. Let  $\sigma_i^K(p)$  be the smallest price of good  $i$  in any profile on any improvement path such that only players in the set  $K$  move on this path. Formally, let the set  $\Phi^K(p) \subseteq \mathcal{P}(p)$  be the set of profiles such that for any profile  $p' \in \Phi^K(p)$  there is an improvement path  $\rho \in \mathcal{P}(p)$  from  $p$  to  $p'$  that respects the scheduler  $f$ , and on which for any  $\hat{p} \in \rho$ ,  $\hat{p} \neq p'$ , the deviating player  $f(\hat{p}) \in K$ . The  $\sigma_i^K(p)$  is then defined as

$$\sigma_i^K(p) = \min_{p' \in \Phi^K(p)} p_i'^S.$$

We will use several expressions constructed in this manner. In particular  $\sigma_i^S(p)$  is then the smallest price of good  $i$  obtained after all sellers move from profile  $p$ . In practice, this value either equals the highest bid for the good, or the current seller’s minimum price level if there are no bids above the cost. Another useful expression is  $\sigma_i^{S \cup T(p)}(p)$ , the smallest price of good  $i$  after the sellers and the top bidders move, because for the prices to decrease a player from  $S \cup T(p)$  has to move.

Next, let  $\phi_j^K(p)$  be the highest payoff that buyer  $j \in B$  can obtain at current prices in any profile on any improvement path such that only players in the set  $K$  move on

this path. Formally we can define  $\phi_j^K$  as follows:

$$\phi_j^K(p) = \max_{p' \in \Phi^K(p)} \max_{i \in S} (h_{ij} - \sigma_i^S).$$

The value of  $\phi_j(p)$  represents the opportunity for buyer  $j$  to gain by switching the good that she is bidding for assuming that any ties would break in her favor. In other words, it is the highest difference between the value of any good and the price  $\sigma_i^S$ , which is either the highest bid or the seller's minimum price, whichever is higher. These variables are a convenient way to simplify proofs of potentials for schedulers; by definition, the value of  $\phi_j(\cdot)$  for any buyer  $j$  can only decrease until some buyer  $j' \notin K$  moves (see Lemma 1). Notice that we do not require improvement paths to be maximal, and in particular the unique path in  $\Phi^K(p)$  could be a singleton  $\rho = (p)$ , e.g. if  $f(p) \notin K$ . Moreover, such path always exists,  $\Phi_j^K(p)$  is nonempty and therefore  $\phi_j^K(p)$  and  $\sigma_i^K(p)$  are well-defined.

The "component of potential"  $F_k^{BR}(p)$  for a buyer  $k$  is the difference between  $\phi_k^{S \cup \{j \in B: j \preceq_f k\}}(p)$  and the payoff implied by the buyer's current bid if she were the unique top bidder regardless of whether she is actually winning, is in a tie or is not the highest bidder. If this difference is negative,  $F_k^{BR}(p) = 0$ . The component for the seller is just the difference between the highest bid and the seller's price, the "spread". In other words, the expression for  $F_k^{BR}(p)$  is then:

$$F_k^{BR}(p) = \begin{cases} |p_k^S - \max_{j \in B} p_{kj}| & \text{if } k \in S \\ \max(0, \phi_k^{S \cup \{j \in B: j \preceq_f k\}}(p) - (h_{m(k)k} - p_{m(k)k}^B)) & \text{if } k \in B. \end{cases}$$

A similar expression  $F_k(p)$  for better responses is the difference between  $\phi_k^{T(p) \cup S}(p)$  and the payoff implied by the buyer's current bid:

$$F_k(p) = \begin{cases} |p_k^S - \max_{j \in B} p_{kj}^B| & \text{if } k \in S \\ \max(0, \phi_k^{T(p) \cup S}(p) - (h_{m(k)k} - p_{m(k)k}^B)) & \text{if } k \in B. \end{cases}$$

Notice that the only changes from the best-response case is that we consider the whole set of top bidders and sellers  $T(p) \cup S$  instead of the set of players that precede  $k$  in  $\prec_f$ .

These definitions require some explanation. The dynamics of the game involve two types of behavior: buyers shifting between goods for higher payoff thus lowering the  $F_k(\cdot)$  or  $F_k^{BR}(\cdot)$  and bidding wars with generally zero  $F_k(\cdot)$  and  $F_k^{BR}(\cdot)$ . The functions will allow us to separate the behaviors and ensure that shifting between goods precede the bidding wars thus ensuring that there are no cycles.

The following is the only new definition introduced in this paper, and we use it to more concisely describe the potentials for schedulers:

**Definition 1.** *We will say that a function  $\beta$  is **nondecreasing (nonincreasing) in the better responses** of a player  $k \in K$  or of the set of players  $K$  at some profile  $p$  if  $\beta(p) = \min_{p' \in \Phi^K(p)} \gamma(p')$  (or  $\beta(p) = \max_{p' \in \Phi^K(p)} \gamma(p')$ ) for some function  $\gamma : \Omega \rightarrow \mathbb{R}$ .*

The expressions  $\phi_j^K(p)$  introduced above are non-increasing in better responses of players in the set  $K$ , while  $\sigma_i^K(p)$  are non-decreasing in their better responses. The key property of such functions is that the value is non-increasing or non-decreasing until a player that is not in  $K$  moves. This is captured by the following lemma.

**Lemma 1.** *Take an action profile  $p$  and any profile  $p'$  that is an improvement for  $f(p)$  from  $p$ . If function  $\beta$  is non-decreasing in the better responses of the set of players  $K$*

at  $p$  and  $f(p) \in K$  then  $\beta(p) \leq \beta(p')$ . If  $\beta$  is non-increasing in the better responses of the set of players  $K$  at  $p$  and  $f(p) \in K$  then  $\beta(p) \geq \beta(p')$ .

*Proof.* Since  $f(p) \in K$ , for any improvement path  $\rho' \in \Phi^K(p')$  there is also an improvement path  $\rho = (p) \frown \rho' \in \Phi^K(p)$ , obtained by prepending  $p$  to  $\rho'$ . Since  $\beta(\cdot)$  is the maximum (or minimum) over these sets,  $\beta(p) \geq \beta(p')$  (or  $\beta(p) \leq \beta(p')$ ).  $\square$

We now deal with the problem of tie-breaking. For this we will introduce a function  $\mathfrak{C}(\cdot)$ :

$$\mathfrak{C}(p) = \sum_{j \in T(p)} \sum_{\substack{j' \in T(p): \\ m_{j'}(p) \neq m_j(p)}} \pi_{m_j(p)j} \pi_{m_{j'}(p)j'}$$

. The  $\mathfrak{C}$  stands for ‘‘congestion’’, because the part of the improvement path where players shift between tied outcomes resembles a congestion game. Similarly to  $\sigma$  and  $\phi$  variables above, let  $\tau^K(p) = \min_{p' \in \Phi^K(p)} \mathfrak{C}(p')$  for changes in  $\mathfrak{C}$  when players in  $K$  move.

**A.2. Proofs of Propositions 1 and 2.** For the proofs of the main results we will separately consider the bidders who are buying some good  $T(p)$  and the rest of the buyers  $B \setminus T(p)$ . The next lemma deals with the former, simpler case when the deviating buyer is buying some good in  $p$ . Since the move to  $p'$  is an improvement, she has to be buying some good in  $p'$  as well, i.e.  $k \in T(p)$  and then  $k \in T(p')$ .

**Lemma 2.** *Take a scheduler  $f$  with  $S \prec_f B$ , an action profile  $p$ ,  $f(p) = k \in B$  and any profile  $p'$  that is a better response for buyer  $k$  from  $p$ . Suppose also that  $k \in T(p)$  and thus also  $k \in T(p')$ . Then*

- (i) *If  $h_{ik} - p_{ik}^B < h_{i'k} - p_{i'k}^B$  then both  $F_k^{BR}(p) > F_k^{BR}(p')$  and  $F_k(p) > F_k(p')$ ,*
- (ii) *If  $h_{ik} - p_{ik}^B \geq h_{i'k} - p_{i'k}^B$  then  $\mathfrak{C}(p) < \mathfrak{C}(p')$ .*

*Proof.* Let  $m_k(p) = i$  and  $m_k(p) = i'$ . Since by construction  $\phi_k^{S \cup \{j \in B, j \preceq_f k\}}(\cdot)$  is non-increasing in better responses of  $k \in S \cup \{j \in B, j \preceq_f k\}$ , it does not increase in  $p'$

by Lemma 1. Similarly, since by construction  $\phi_k^{T(p) \cup S}(p)$  is non-increasing in better responses of  $k \in T(p) \cup S$ , it does not increase in  $p'$  by Lemma 1 as well. We must then have  $F_k^{BR}(p) \geq F_k^{BR}(p')$  and  $F_k(p) \geq F_k(p')$ . We now consider the two cases separately.

- (i) Suppose first that  $h_{ik} - p_{ik}^B < h_{i'k} - p_{i'k}^B$ . Then  $\phi_k^{S \cup \{j \in B, j \preceq_f k\}}(p) \geq \phi_k^{S \cup \{j \in B, j \preceq_f k\}}(p') \geq h_{i'k} - p_{i'k}^B > h_{ik} - p_{ik}^B$ , and thus  $F_k^{BR}(p) > 0$  and  $F_k^{BR}(p) > F_k^{BR}(p')$ . For better responses  $F_k(p) > F_k(p')$  follows by the same argument.
- (ii) Suppose now instead that  $h_{ik} - p_{ik}^B \geq h_{i'k} - p_{i'k}^B$ . Note that  $U_k(p') = \frac{\pi_{i'k}(p)}{\sum_{j \in T_{i'}} \pi_{i'j}}(h_{i'k} - p_{i'k}^B)$  and  $U_k(p) = \frac{\pi_{ik}(p)}{\sum_{j \in T_i} \pi_{ij}}(h_{ik} - p_{ik}^B)$ . For  $p'$  to be a better response for  $k$  we must have  $U_k(p) < U_k(p')$ , and thus  $\frac{\pi_{ik}(p)}{\sum_{j \in T_i} \pi_{ij}} < \frac{\pi_{i'k}(p)}{\sum_{j \in T_{i'}} \pi_{i'j}}$  or  $\pi_{ik}(p) \sum_{j \in T_{i'}} \pi_{i'j} < \pi_{i'k}(p) \sum_{j \in T_i} \pi_{ij}(p)$ . At the same time

$$\mathfrak{C}(p') - \mathfrak{C}(p) = 2 \left( \pi_{i'k}(p) \sum_{j \in T_i} \pi_{ij} - \pi_{ik}(p) \sum_{j \in T_{i'}} \pi_{i'j} \right),$$

which is therefore positive and so  $\mathfrak{C}(p) < \mathfrak{C}(p')$  as required. □

The next lemmas will be useful for the other case when the deviating buyer was not buying a good in  $p$ . This is the "on-path" part of the proof for the best-response sequence that behaves similarly to the auction in [Demange et al. \(1986\)](#).

**Lemma 3.** *Take a scheduler  $f$  with  $S \prec_f B$ , and an action profile  $p$ ,  $f(p) = k \in B \setminus T(p)$ . Then in any profile  $p'$  that is a better response for  $k$  in  $p$ :*

- (i)  $\sigma_i^S(p') \geq \sigma_i^S(p)$  for any  $i \in S$   
(ii) if  $\sigma_i^S(p') = \sigma_i^S(p)$  for all  $i \in S$  then  $\mathfrak{C}(p) < \mathfrak{C}(p')$ .

*Proof.* Either there is some other buyer  $j' \in B$  who bids more than  $k$  in  $p$  for  $m_k(p)$  with  $p_{m_k(p)j'}^B \geq p_{m_k(p)k}^B$  or there is no trade with  $p_{m_k(p)}^S > p_{m_k(p)j}^B$  and  $\max_{j \in B} p_{m_k(p)j}^B \leq$

$c_{m_k(p)}$ . Then  $k$  either became the highest bidder with a higher bid, the highest bidder  $j'$  is unchanged, or the good is not sold. In any of these cases if the seller of the good has a deviation, it is to increase the price. Therefore, (i) holds.

Moreover, if  $k$  matched the current price  $p_i^S = \sigma_i^S(p)$ , and thus  $\sigma_i^S(p') = \sigma_i^S(p)$  for all  $i \in S$ , then (ii) must be true by definition of  $\mathfrak{C}$  since  $T(p') = T(p) \cup k$ , i.e. the set  $T(p)$  has expanded while no winning buyers changed for other goods.  $\square$

**Lemma 4.** *Take a scheduler  $f$  with  $S \prec_f B$ . Suppose also that  $f(p) = j \in B$  and let  $p'$  be an improvement for  $j$  from  $p$  that respects  $f$ . If  $\sigma_i^S(p) > \sigma_i^S(p')$  for some  $i \in S$  then  $\max_{i \in S} (h_{ij} - \sigma_i^S) > h_{m_j(p)j} - p_{m_j(p)j}^B$ .*

*Proof.* For any seller  $i \in S$  to decrease the price in some state after  $p'$ , the highest bid for her good has to be above the cost but below  $p_i'^S$ . Since  $f(p) = j \in B$ ,  $\sigma_i^S(p) = p_i^S$  and either  $\max_{j' \in B} p_{ij'}^B < p_i'^S = p_i^S = \max_{j' \in B} p_{ij'}^B$ , or  $\max_{j' \in B} p_{ij'}^B \leq c_i < \max_{j' \in B} p_{ij'}^B$ . In other words, the highest bid should have decreased, or some buyer was the first to offer a bid above the seller's cost. The latter case cannot be an improvement for the buyer  $j$  since the good is not sold and her payoff is zero. Therefore,  $\max_{j' \in B} p_{ij'}^B < \max_{j \in B} p_{ij}^B$ . Then the buyer  $j$  was the unique top bidder for  $i$  and  $\max_{i \in S} (h_{ij} - \sigma_i^S) \geq U_j(p') > U_j(p) = h_{m_j(p)j} - p_{m_j(p)j}^B$ .  $\square$

The next lemma shows that if no non-tied player wants to switch good in some set of players  $K$ , then prices and bids will continue rising until an equilibrium is reached or a player not in  $K$  moves. This effectively separates the adjustments of top bidders from the bidding wars.

**Lemma 5.** *Take an action profile  $p$  and a scheduler  $f$  with  $S \prec_f B$ . Suppose  $\max_{i \in S} (h_{ij} - \sigma_i^S) \leq h_{m_j(p)j} - p_{m_j(p)j}^B$  for any  $j$  in some set of buyers  $K$ . Then in any state  $p'$  such that there is an improvement path  $\rho = (p, \dots, p') \in \mathcal{P}(p)$  that respects  $f$  and such that  $f(\hat{p}) \in K$  for all  $\hat{p} \in \rho$ ,  $\hat{p} \neq p'$ :*

- (i) For any  $i \in S$ ,  $\sigma_i^S(p) \leq \sigma_i^S(p')$ ,
- (ii)  $\max(0, \max_{i \in S}(h_{ij} - \sigma_i^S(p))) \geq \max(0, \max_{i \in S}(h_{ij} - \sigma_i^S(p')))$  for any  $j \in B, j \preceq_f k$ ,
- (iii)  $\max_{i \in S}(h_{ij} - \sigma_i^S(p')) \leq (h_{m(j)j} - p_{m(j)j}^{lB})$  for any  $j \in B, j \preceq_f k$ ,

*Proof.* By induction. The plan for the proof that the three conditions hold for any profile  $p' \in \rho$ , with every condition implying the next.

Let  $p^1 = p$ , and notice that the three conditions above hold for  $p^1$ . Suppose the lemma is true for all profiles in  $\rho = (p^1, p^2, \dots)$  up to some  $p^l$ . We will show that the lemma holds for the next profile  $p^{l+1}$  as well, if such profile exists. If we had  $k \notin K$  then  $p^l$  would have been the last profile in  $\rho$ . If instead  $f(p^l) \in S \cap K$  then  $\sigma_i^S(p) \leq \sigma_i^S(p^l)$  by construction. The remaining case is  $f(p^l) \in B \cap K$ . By Lemma 4 since (iii) holds for  $p^l$ , (i) follows for  $p^{l+1}$ .

It is easy to see that from (i) it follows that

$$\max(0, \max_{i \in S}(h_{ij} - \sigma_i^S(p^{l+1}))) \leq \max(0, \max_{i \in S}(h_{ij} - \sigma_i^S(p^l)))$$

for any  $j \in B \cap K$ . That is, the best possible trade (ignoring the ties) is no better in  $p^{l+1}$  than in  $p^l$  for any  $j \in B$ . This implies part (ii).

For  $j = f(p^l)$  we must have  $\max_{i \in S}(h_{ij} - \sigma_i^S(p^{l+1})) \leq (h_{m(j)j} - p_{m(j)j}^{(l+1)B})$ , otherwise she is not playing the best response in  $p^{l+1}$ . For any other buyer  $j \in K \cap B, j \neq f(p^l)$  the action has not changed, i.e.  $p_j^{lB} = p_j^{(l+1)B}$  and thus by (ii) for these players we also have  $\max_{i \in S}(h_{ij} - \sigma_i^S(p^{l+1})) \leq \max_{i \in S}(h_{ij} - \sigma_i^S(p^l) \leq (h_{m(j)j} - p_{m(j)j}^{lB}) = (h_{m(j)j} - p_{m(j)j}^{(l+1)B})$  for any  $j \in K \cap B$ . By induction this implies (iii). □

We are now ready to prove Proposition 1.

*Proof of Proposition 1.* The goal of the proof is to introduce an (incomplete) acyclic total ordering  $\triangleleft^{BR}$  such that the states in any improvement path that respects a local

scheduler  $f$  can be ordered by  $\triangleleft^{BR}$ . Define  $\triangleleft^{BR}$  as the following lexicographic ordering on admissible action profiles  $p$ :  $p' \triangleleft^{BR} p$  if either

$$(a) \quad \begin{cases} F_{j'}^{BR}(p) > F_{j'}^{BR}(p') & \text{for some } j' \in B, \\ F_j^{BR}(p) = F_j^{BR}(p') & \text{for all } j \in B, j \prec_f k, \end{cases}$$

or

$$(b) \quad \begin{cases} F_j^{BR}(p) = F_j^{BR}(p') & \text{for all } j \in B, \\ \sigma_i^S(p) \leq \sigma_i^S(p') & \text{for all } i \in S, \\ \sigma_{i'}^S(p) < \sigma_{i'}^S(p') & \text{for some } i' \in S, \end{cases}$$

or

$$(c) \quad \begin{cases} F_j^{BR}(p) = F_j^{BR}(p') & \text{for all } j \in B, \\ \sigma_i^S(p) = \sigma_i^S(p') & \text{for all } i \in S, \\ \mathfrak{C}(p) < \mathfrak{C}(p') \end{cases}$$

or

$$(d) \quad \begin{cases} F_j^{BR}(p) = F_j^{BR}(p') & \text{for all } j \in B, \\ \sigma_i^S(p) = \sigma_i^S(p') & \text{for all } i \in S, \\ \mathfrak{C}(p) = \mathfrak{C}(p') \\ F_{i'}^{BR}(p) > F_{i'}^{BR}(p') & \text{for some } i' \in S, \\ F_{\hat{i}}^{BR}(p) = F_{\hat{i}}^{BR}(p') & \text{for all } \hat{i} \in S, \hat{i} \prec_f i'. \end{cases}$$

That is, the profiles are sorted according to components for buyers  $F_j^{BR}$ , prices  $\sigma_i$ , tie-breaking component  $\mathfrak{C}$ , and sellers' components  $F_i^{BR}$  in this order of importance.

Consider a BR-improvement path  $\rho = (\dots, p, p')$  that respects  $f$ . We need to show that  $p \triangleleft^{BR} p'$ .

Suppose first that the improving player is a seller,  $f(p) = k \in S$ . In any profile  $p$  where non-zero payoff for  $k$  is possible she has a unique best response  $p_k^S = \max_{j \in B} p_{ij}^B$ . Since  $\phi_j(\cdot)$  is non-increasing in sellers' better responses for any buyer  $j \in B$ , we have  $\phi_j(p) \geq \phi_j(p')$  for all  $j \in B$  by Lemma 1. In turn, since the seller's price only enters  $F_{k'}^{BR}(\cdot)$  through  $\phi$ , this implies that  $F_{k'}^{BR}(p') \leq F_{k'}^{BR}(p)$  for all  $k' \in B \cup (S \setminus f(p))$ . Since  $\sigma_i^S$  for any  $i \in S$  is non-decreasing in seller  $k$ 's responses by Lemma 1,  $\sigma_i^S(p) \leq \sigma_i^S(p')$ . Moreover, since  $p'$  must be an improvement,  $T_k(p') \subseteq T_k(p)$  and thus  $\mathfrak{C}(p) \leq \mathfrak{C}(p')$ . For seller  $k$  to have a better response at  $p$  it is necessary that  $p_k^S \neq \max_{j \in B} p_{kj}^B$ . Thus,  $F_k^{BR}(p) > F_k^{BR}(p')$  and, either by part (d) or by one of the conditions in the other parts,  $p' \triangleleft^{BR} p$ .

Now suppose that the improving player is a buyer,  $f(p) = k \in B$ . By construction  $\phi_j^{S \cup \{j' \in B: j' \preceq_f j\}}$  of any  $j \in B, k \prec_f j$  is non-increasing in better responses of  $k$  and  $\phi_j^{S \cup \{j' \in B: j' \preceq_f j\}}(p') \leq \phi_j^{S \cup \{j' \in B: j' \preceq_f j\}}(p)$  by Lemma 1. Moreover, since  $p_j^B = p_j^B$  for all such  $j \in B, k \prec_f j$  we also have  $F_j^{BR}(p) \geq F_j^{BR}(p')$ . Note also that for the new action profile to be a better response, it must be that  $k \in T(p')$ .

We will continue with two cases:

- (1) Suppose buyer  $k$  does not have the winning bid in  $p$ , that is  $k \in B \setminus T(p)$ . Then in  $p'$  it must be that  $\max_{i \in S} (h_{ik} - \sigma_i^S(p')) \leq (h_{m(k)k} - p_{m(k)k}^B)$  because  $k$  played a best response. At the same time since  $f(p) = k$ ,  $\max_{i \in S} (h_{ij} - \sigma_i^S(p)) \leq (h_{m(j)j} - p_{m(j)j}^B)$  for any  $j \in B, j \preceq_f k$ . Moreover, since at  $p'$  all bids are the same and buyer  $k$ 's bid is no less than in  $p$ ,  $\sigma_i^S(p') \geq \sigma_i^S(p)$  and thus  $\max_{i \in S} (h_{ij} - \sigma_i^S(p')) \leq (h_{m(j)j} - p_{m(j)j}^B)$  for any  $j \in B, j \preceq_f k$ . Thus, because  $\Phi^{S \cup \{j' \in B: j' \preceq_f j\}} \subseteq \Phi^{S \cup \{j' \in B: j' \preceq_f k\}}$  for any  $j \preceq_f k$ , part (iii) of Lemma 5 for  $K = S \cup \{j' \in B : j' \preceq_f k\}$  implies  $F_j^{BR}(p') = 0$  for any  $j \preceq_f k$ . If

$F_j^{BR}(p) > F_j^{BR}(p')$  for some  $j \preceq_f k$  then  $p' \triangleleft^{BR} p$  by part (a). Suppose instead that  $F_j^{BR}(p) = F_j^{BR}(p')$  for all such  $j$ . Let the current highest bidder for the good  $m_k(p')$  be  $j' \in B$ . By Lemma 3 one of the two cases must occur in  $p'$ . Either buyer  $k$  bids higher than  $j'$  and the highest bid for that good is higher in  $p'$  or she bids the same as  $j'$ , but  $\mathfrak{C}(p) < \mathfrak{C}(p')$ . In both cases either by part (b) or by part (c) again  $p' \triangleleft^{BR} p$ .

- (2) Suppose now instead that buyer  $k$  is buying both in  $p$  and  $p'$ , that is  $k \in T(p), k \in T(p')$ . By Lemma 4, if  $\sigma_i^S(p') < \sigma_i^S(p)$  for some  $i \in S$  then  $\max_{i \in S}(h_{ik} - \sigma_i^S(p)) > h_{m_k(p)j} - p_{m_k(p)j}^B$  and then since  $k$  has to play a best response in  $p'$ ,  $\max_{i \in S}(h_{ik} - \sigma_i^S(p')) = h_{m_k(p')j} - p_{m_k(p')j}^B > h_{m_k(p)j} - p_{m_k(p)j}^B$ . However, then by Lemma 2,  $F_k^{BR}(p') > F_k^{BR}(p)$  and by part (a),  $p' \triangleleft^{BR} p$ . If this is not the case then  $\sigma_i^S(p') \geq \sigma_i^S(p)$  and by Lemma 2,  $F_k^{BR}(p') < F_k^{BR}(p)$  or  $\mathfrak{C}(p) < \mathfrak{C}(p')$  and, by part (a), part (b) or part (c),  $p' \triangleleft^{BR} p$ .

Therefore, for any  $\rho$  that respects  $f$  we have  $p' \triangleleft^{BR} p$ . Finally since by construction  $\triangleleft^{BR}$  is acyclic, there is an associated  $f$ -potential function  $F : F(p') < F(p)$  iff  $p' \triangleleft^{BR} p$ . This in turn implies that any market game respects every such local scheduler  $f$  by Theorem 9 in Apt and Simon (2015). □

We now prove the result for better responses. All lemmas above apply to the better-response case. The only effect that is not present is that the  $F_k(p)$  is not necessarily zero after an improvement by  $k \in B \setminus T(p)$ , because it does not have to be the best response. This fact is used in the proof of Proposition 1 above. For better responses we ensure that  $f$  schedules  $k$  to move again until she reaches the best response.

*Proof of Proposition 2.* Let the set  $\bar{T}(p) \subseteq T(p)$  for any profile  $p$  include all buyers that are in  $T(p)$  and have  $\max_{i \in S}(h_{ik} - \sigma_i^S(p)) > (h_{m(k)k} - p_{m(k)k}^B)$ . Take a scheduler  $f$  that prioritizes players in this order:  $S \prec_f \bar{T}(\cdot) \prec_f T \setminus \bar{T}(\cdot) \prec_f B \setminus T(\cdot)$

Define  $\triangleleft$  as the following lexicographic ordering on admissible action profiles  $p$ :  
 $p' \triangleleft p$  if either

$$(a) \quad \begin{cases} \sigma_i^{\bar{T}US}(p) \leq \sigma_i^{\bar{T}US}(p') & \text{for all } i \in S, \\ \sigma_{i'}^{\bar{T}US}(p) < \sigma_{i'}^{\bar{T}US}(p') & \text{for some } i' \in S, \end{cases}$$

or

$$(b) \quad \begin{cases} \sigma_i^{\bar{T}US}(p) = \sigma_i^{\bar{T}US}(p') & \text{for all } i \in S, \\ \tau^{\bar{T}US}(p) < \tau^{\bar{T}US}(p'), \end{cases}$$

or

$$(c) \quad \begin{cases} \sigma_i^{\bar{T}US}(p) = \sigma_i^{\bar{T}US}(p') \text{ and} \\ \tau^{\bar{T}US}(p) = \tau^{\bar{T}US}(p') \\ \sum_{j \in B} F_j(p) > \sum_{j \in B} F_j(p') \end{cases},$$

or

$$(d) \quad \begin{cases} \sigma_i^{\bar{T}US}(p) = \sigma_i^{\bar{T}US}(p') \text{ and} \\ \sum_{j \in B} F_j(p) = \sum_{j \in B} F_j(p') \\ \tau^{\bar{T}US}(p) = \tau^{\bar{T}US}(p') \\ \mathfrak{C}(p) < \mathfrak{C}(p') \end{cases},$$

or

$$(e) \quad \left\{ \begin{array}{ll} \sigma_i^{\bar{T} \cup S}(p) = \sigma_i^{\bar{T} \cup S}(p'), \text{ and} \\ \sum_{j \in B} F_j(p) = \sum_{j \in B} F_j(p'), \\ \tau^{\bar{T} \cup S}(p) = \tau^{\bar{T} \cup S}(p') & , \\ \mathfrak{C}(p) = \mathfrak{C}(p') & , \\ F_{i'}(p) > F_{i'}(p') & \text{for some } i' \in S, \\ F_{\hat{i}}(p) = F_{\hat{i}}(p') & \text{for all } \hat{i} \in S, \hat{i} \prec_f i'. \end{array} \right.$$

That is, the profiles are sorted according to prices  $\sigma_i^{\bar{T} \cup S}$ , tie-breaking component  $\tau^{\bar{T} \cup S}$  that is non-decreasing in better responses of  $\bar{T} \cup S$ , the sum  $\sum_{j \in B} F_j$ , tie-breaking component  $\mathfrak{C}$ , and sellers' components  $F_i^{BR}$  in this order of importance.

Consider a BR-improvement path  $\rho = (\dots, p, p')$  that respects  $f$ . We need to show that  $p \triangleleft^{BR} p'$ .

Suppose first that the improving player is a seller,  $f(p) = k \in S$ . Since the sellers again move first according to  $f$ , the argument for  $k \in S$  is unchanged from Proposition 1. In any profile  $p$  where non-zero payoff for  $k$  is possible, a seller has a unique best response  $p_k^S = \max_{j \in B} p_{ij}^B$ . Since  $\phi_j(\cdot)$  is non-increasing in sellers' better responses for any buyer  $j \in B$ , we have  $\phi_j(p') \leq \phi_j(p)$  for all  $j \in B$  by Lemma 1. In turn, since the seller's price only enters  $F_{k'}(\cdot)$  through  $\phi$ , this implies that  $F_{k'}(p') \leq F_{k'}(p)$  for all  $k' \in B \cup (S \setminus f(p))$ . Since  $\sigma_i^{\bar{T} \cup S}$  and  $\tau_i^{S \cup \bar{T}}$  are non-decreasing in sellers' responses by Lemma 1,  $\sigma^{\bar{T} \cup S} \leq \sigma^{\bar{T} \cup S}$  and  $\tau^{\bar{T} \cup S}(p) \geq \tau^{\bar{T} \cup S}(p')$ . Moreover, since  $p'$  must be an improvement,  $T_k(p') \subseteq T_k(p)$  and thus  $\mathfrak{C}(p) \leq \mathfrak{C}(p')$ . For seller  $k$  to have a better response it is necessary that  $p_k^S(p') \neq \max_{j \in B} p_{kj}^B(p')$ . Thus,  $F_k(p') < F_k(p)$ , and, therefore either by part (e) or by one of the conditions in the other parts,  $p' \triangleleft^{BR} p$ .

Now suppose that the improving player is a buyer, that is  $f(p) = k \in B$ .

Note again that for the new action profile to be a better response, it must be that  $k \in T(p')$ .

We will continue the proof by cases:

(1) Suppose buyer  $k$  does not have the winning bid in  $p$ , that is  $k \in B \setminus T(p)$ .

Since  $f(p) = k$ ,  $\max_{i \in S}(h_{ij} - \sigma_i^S(p)) \leq (h_{m(j)j} - p_{m(j)j}^B)$  for any  $j \in T(p) \setminus k$ .

Moreover, since at  $p'$  all bids are the same and buyer  $k$ 's bid is no less than in  $p$ ,  $\sigma_i^S(p) \leq \sigma_i^S(p')$  and thus  $\max_{i \in S}(h_{ij} - \sigma_i^S(p')) \leq (h_{m(j)j} - p_{m(j)j}^B)$  for any  $j \in T(p) \setminus k$ . Let  $p^1$  be the first state after  $p$  with  $f(p) \notin S$ . Then in  $p^1$  either  $\bar{T}(p) = \{k\}$  and  $f(p^1) = k$  or  $\bar{T}(p) = \emptyset$ . In the first case  $k$  moves again.

Player  $k$  continues to move until  $\max_{i \in S}(h_{ik} - \sigma_i^S(\bar{p})) \leq (h_{m(k)k} - \bar{p}_{m(k)k}^B)$  and  $\bar{T}(p) = \emptyset$  and the sellers finish moving. This has to happen eventually

in some state  $\bar{p}$  because by Lemma 2 either  $(h_{m(k)k} - \bar{p}_{m(k)k}^B)$  or  $\mathfrak{C}(\cdot)$  increases and both are bounded. At the same time  $\sigma_i^S(p) \leq \sigma_i^S(\bar{p})$  since other bids are unchanged and the bid of  $k$  is no less than the highest bid for  $m_k(\bar{p})$ .

Moreover, if  $\sigma_i^S(p) = \sigma_i^S(\bar{p})$  for every  $i \in S$ , it must be that  $\mathfrak{C}(p) < \mathfrak{C}(\bar{p})$  by Lemma 3. Thus,  $\sigma^{S \cup \bar{T}}(p') = \sigma^S(\bar{p}) \geq \sigma^{S \cup \bar{T}}(p)$  and if it holds with equality then  $\tau^{S \cup \bar{T}}(p') = \mathfrak{C}(\bar{p}) > \tau^{S \cup \bar{T}}(p) = \mathfrak{C}(p)$ . Thus, by part a or b,  $p' \triangleleft^{BR} p$ .

(2) Suppose now instead that buyer  $k$  is buying both in  $p$  and  $p'$ , that is  $k \in T(p), k \in T(p')$ .

By construction  $\phi_j^{S \cup T}$  of any  $j \in B$  is non-decreasing in better responses of  $k$  and  $\phi_j^{S \cup T}(p') \leq \phi_j^{S \cup T}(p)$  by Lemma 1. Moreover, since

$p_{ij}^B = p'_{ij}^B$  for all  $i \in S$  and all  $j \in B \setminus k$  we also have  $F_j(p) \geq F_j(p')$  for any  $j \in B \setminus k$ . If  $k \in \bar{T}(p)$  then, since  $\sigma_i^{S \cup \bar{T}}$  and  $\tau_i^{S \cup \bar{T}}$  are non-decreasing in better responses of  $S \cup \bar{T}$ , by Lemma 1  $\sigma_i^{S \cup \bar{T}}(p) \leq \sigma_i^{S \cup \bar{T}}(p')$  and  $\tau_i^{S \cup \bar{T}}(p) \leq \tau_i^{S \cup \bar{T}}(p')$ .

If instead  $k \in T(p) \setminus \bar{T}(p)$  then by Lemma 4  $\sigma_i^{S \cup \bar{T}}(p) \leq \sigma_i^{S \cup \bar{T}}(p')$  and  $\tau(p) = \mathfrak{C}(p)$ .

As  $\sigma^S(p) \leq \sigma^S(p')$ ,  $\bar{T}(p') = \{\emptyset\}$  and therefore  $\tau(p') = \mathfrak{C}(p')$ . At the same time

by Lemma 2,  $F_k^{BR}(p') < F_k^{BR}(p)$  or  $\mathfrak{C}(p) < \mathfrak{C}(p')$ . Thus, either by part (c) or part (d) or by other parts,  $p' \triangleleft^{BR} p$ .

Therefore, for any  $\rho$  that respects  $f$  we have  $p' \triangleleft p$ . Finally since by construction  $\triangleleft$  is acyclic, there is an associated  $f$ -potential function  $F : F(p') < F(p)$  iff  $p' \triangleleft p$ . This in turn implies that any market game respects every such state scheduler  $f$  by Theorem 9 in [Apt and Simon \(2015\)](#).

□

## APPENDIX B. CHARACTERIZATION OF EQUILIBRIA

In this appendix we present the characterization of the "sinks" of the convergence process, i.e. the Nash equilibria of the market game.

**B.1. Competitive equilibria.** While we do not focus on competitive behavior, we will use competitive equilibria to characterize and narrow down the set of Nash equilibria.

A *competitive equilibrium* for the economy  $(h, c)$  is a pair  $(y, \tilde{p})$ , where

$$y = ((y_i)_{i \in S}, (y_j)_{j \in B}) \in \prod_{i \in S} Y_i \times \prod_{j \in B} Y_j$$

and  $\tilde{p} = (\tilde{p}_i)_{i \in S} \in \mathbb{R}_+^M$  such that

$$(1) \quad \begin{aligned} \max_{i \in S} (y_{ij} h_{ij}) - \sum_{i \in S} \tilde{p}_i y_{ij} &\geq \max_{i \in S} (y'_{ij} h_{ij}) - \sum_{i \in S} \tilde{p}_i y'_{ij} \\ (\tilde{p}_i - c_i) y_i &\geq (\tilde{p}_i - c_i) y'_i \text{ for all } y'_i \in Y_i, \text{ for all } i \in S, \text{ and} \\ \sum_{j \in B} y_{ij} &= y_i \text{ for all } i \in S. \end{aligned}$$

The first set of conditions represent utility maximization by buyers and encode the assumption that buyers can enjoy at most one good. The second set of conditions

represent profit maximization by the sellers. The third set of conditions includes the market clearing conditions for each of the goods.

**B.2. Nash equilibria.** To describe the set of Nash equilibria of the market game, note that for every subset  $S' \subseteq S$  (including the empty set), we can define a subeconomy  $(h', c')$  for goods and sellers  $S'$  and buyers  $B$ . Let  $(y', \tilde{p}')$  be a competitive equilibrium for any such economy, where

$$y' = ((y'_i)_{i \in S'}, (y'_j)_{j \in B}) \in Y_{S', B} \equiv \prod_{i \in S'} Y_i \times \prod_{j \in B} Y_j \quad \text{and} \quad \tilde{p}' = (\tilde{p}'_i)_{i \in S'} \in \mathbb{R}_+^{|S'|}$$

With a slight abuse of notation, for any  $y \in Y_{S', B}$ , let

$$x(y) \equiv (x \in X : x_{ij} = y'_{ij} \text{ if } i \in S' \text{ and } j \in B, \text{ and } x_{i'j} = 0 \text{ if } i' \notin S' \text{ and } j \in B).$$

Note that if  $x(y') \in F(p)$  for some bid profile  $p = (p_1^B, \dots, p_N^B, p_1^S, \dots, p_M^S)$  such that

$$y'_{i'j} = 1 \quad \Rightarrow \quad p_i^S = p_{ij}^B = \tilde{p}'_i \quad \text{for every } i' \in S' \text{ and } j \in B,$$

then  $x(y')$  induces the same allocation than the competitive equilibrium  $(y', \tilde{p}')$ ; that is, it induces the same assignment and the same money transfers than  $(y', \tilde{p}')$  for every  $i \in S'$  and  $j \in B$ , with any other seller  $i' \notin S'$  left unassigned.

We say that a bid profile  $p$  induces the same allocation than  $(y', \tilde{p}')$  if  $\Pi(p)$  is a singleton satisfying  $\Pi(p) = \{x(y')\}$ , and moreover, for every  $i' \in S'$  and  $j \in B$  such that  $y'_{i'j} = 1$ , we have  $p_i^S = p_{ij}^B = \tilde{p}'_i$ .

The theorems in this section show that the Nash equilibria generally correspond to competitive equilibria of some subeconomy. In other words, a Nash equilibrium is either a competitive equilibrium for the whole economy or a competitive equilibrium provided that some markets fail to open due to miscoordination. This largely mirrors the semi-Walrasian equilibria in [Mas-Colell \(1982\)](#). The only complication is caused by the tie-breaks that slightly expand the set of Nash equilibria in terms of prices.

More precisely, the set of matchings resulting from Nash equilibria is the same as the set of matchings of competitive equilibria for some subeconomy, but this result requires our Assumption 2 for coarse space. The Nash equilibrium prices don't have to be competitive, but are always within  $\epsilon = 1$  from them.

We start with the “competitive  $\implies$  Nash” direction, which is straightforward. This and the other results in this section are independent of tie-breaking rules. In particular, note that  $\Pi(p)$  is a singleton below.

**Theorem 1.** *Let  $(y', \tilde{p}')$  be a competitive equilibrium for some economy  $(h', c)$  consisting of goods  $S' \subseteq S$  and such that  $\tilde{p}' \in \mathbb{Z}^M$ . Then there is a bid profile  $p$  that induces the same allocation, prices, and payoffs in (possibly one of many) Nash equilibria of the strategic market game for the larger economy  $\{h, c\}$ .*

*Proof.* To prove the theorem, consider  $p$  such that

$$p_i^S = \begin{cases} \tilde{p}'_i & \text{if } i \in S' \\ \kappa & \text{if } i \notin S' \end{cases}$$

for some  $\kappa > \max_{i \in S, j \in B} h_{ij}$ , and

$$p_{ij}^B = \begin{cases} \tilde{p}'_i & \text{if } y'_{ij} = 1 \\ 0 & \text{otherwise} \end{cases} .$$

While the assignment  $x(y')$  makes zero surplus, every other assignment makes zero or negative surplus, and moreover the assignment above ray-dominates every other zero-surplus assignment. Hence,  $\Pi(p) = \{x(y')\}$ , as desired.

It is straightforward to check that, since ask prices for goods such that  $i \in S'$  are competitive and prices for goods  $i \notin S'$  are prohibitively expensive, no buyer has an incentive to deviate from  $p$ . Similarly, since bid prices for goods such that  $i \in S'$  are competitive, and prices for goods  $i \notin S'$  are zero, no seller has an incentive to deviate from  $p$ . □

The converse is almost true, i.e. there are no Nash equilibria that are too far away from competitive outcomes. However, the discrete formulation can distort outcomes a little. Without Assumption 2, we can only guarantee competitiveness by adjusting the payoffs by  $\epsilon$ . In particular the following theorem is true with or without Assumption 2.

**Theorem 2.** *Let  $x \in \Pi(p)$  with  $\pi(x) > 0$  and suppose  $p \in \mathbb{Z}^M$  is a Nash equilibrium for a strategic market game for some economy  $\{h, c\}$  with goods  $S$ . Then there is a competitive equilibrium  $(y', \tilde{p}')$  for some subeconomy  $(h^\epsilon, c')$  for goods  $S' \subset S$ , where  $h_{ij}^\epsilon \in \{h_{ij}, h_{ij} + \epsilon\}$  for all  $i \in S', j \in B$  that induces the same allocation as  $x$  with the same prices  $\tilde{p}'_i = p_i^S$  for any  $i \in S'$ .*

*Proof.* Note first that the surplus has to be zero in any Nash equilibrium:

$$\sum_{i \in S} \sum_{j \in B} x_{ij} (p_{ij}^B - p_i^S) = 0 \text{ for all } x \in F(p).$$

If, on the contrary, for some outcome  $x$  and some  $j \in B$  and  $i \in S$ ,  $x_{ij} = 1$  and  $p_{ij}^B > p_i^S$ , then seller  $i$  has a profitable deviation in  $p$  to  $\tilde{p}'_i^S = p_i^S + 1$ . Then  $x$  is still preferable for the clearing house due to the ray-dominance assumption, and in all outcomes in  $p$  seller  $i$  is selling her good for a higher price of  $p_i^S + 1$ . At the same time a negative surplus cannot be chosen by the clearing house, i.e. it is not in  $\Pi(p)$ . Hence,  $x_{ij} = 1$  implies  $p_i^S = p_{ij}^B$ , and every possible match makes a zero surplus.

Let  $S'$  be the subset of sellers whose goods are assigned by  $x$ , let  $y'$  be the solution to  $x(y') = x$ , and let  $\tilde{p}' = (p_i^S)_{i \in S'}$ . We claim that  $(y', \tilde{p}')$  is a competitive equilibrium for the subeconomy  $(h^\epsilon, c')$  for goods  $S'$  and  $h_{ij}^\epsilon = h_{ij} + 1$  if  $x_{ij} = 1$  and  $h_{ij}^\epsilon = h_{ij}$  otherwise.

To see that this is a competitive equilibrium, note first that market clearing is guaranteed by the definition of  $\Pi(p)$ . Profit maximization for each seller  $i$  at the given price  $p_i^S$  is guaranteed by the fact that the dominated strategies  $p_i^S < c_i$  are

disallowed. Moreover, since  $p$  is a Nash equilibrium,  $p_i^S \geq c_i$  so that selling at the price  $p_i^S$  is at least as good as not selling even without this restriction under any market clearing rule.

Finally, we claim that each buyer  $j \in B$  maximizes utility by choosing  $y'_j$  given prices  $\tilde{p}'$ . For suppose not; then one of the following must hold:

- (i)  $x_{ij} = 1$  for some  $i \in S'$  but there is  $i' \in S'$  such that  $h_{i'j}^\epsilon - p_{i'}^S \geq h_{ij}^\epsilon - p_i^S + 1$ ,
- (ii)  $x_{ij} = 0$  for all  $i \in S'$  but there is  $i' \in S'$  such that  $h_{i'j}^\epsilon - p_{i'}^S \geq 1$ .

In case (i), since  $x_{ij} = 1$  we have  $h_{i'j} - p_{i'}^S \geq h_{ij} - p_i^S + 2$  and buyer  $j$  can deviate to  $p_{i'j}^B = p_{i'}^S + 1$  and  $p_{i''j}^B = 0$  for all  $i'' \neq i$ . After the deviation, the clearing house should match  $j$  and  $i'$  since every other match makes zero or negative surplus.

Similarly, in case (ii), buyer  $j$  can deviate to  $p_{i'j}^B = p_{i'}^S$  for some positive reward from a tie. The clearing house should match  $j$  and  $i'$  after the deviation with some positive probability. Finally, since no payoffs have decreased, no players are playing their dominated actions.

Hence, in each of the two cases,  $p$  cannot be a Nash equilibrium. □

This is of course not very satisfying, since we had to pay extra  $\epsilon$  to every matched buyer to ensure competitiveness. Another problem is that without extra assumptions, the Nash equilibrium matchings don't generally have to be competitive. While the matchings are competitive in the perturbed game for  $(h^\epsilon, c')$ , they are not necessarily competitive in the original game for  $(h, c')$ . This is exactly the reason for the Assumption 2.

The following theorem refines the previous result under Assumption 2, stating that all Nash equilibrium matchings are also competitive equilibrium matchings. So the only adverse effect of discrete bids is an  $\epsilon$  change in some prices away from the competitive equilibrium.

**Theorem 3.** *Under Assumption 2, if  $x \in \Pi(p)$  with  $\pi(x) > 0$ , and  $p$  is a Nash equilibrium with  $p^S \in \mathbb{Z}$  for some strategic market game for economy  $\{h, c\}$  with goods  $S$ , then there is a competitive equilibrium  $(y', \tilde{p}^\epsilon)$  for some subeconomy  $(h', c')$  for goods  $S' \subseteq S$  that induces the same allocation as  $x$  with prices  $\tilde{p}_i^\epsilon \in [p_i^S, p_i^S + \epsilon]$  for any  $i \in S'$ .*

*Proof.* Define  $p^\epsilon$  as a vector where for any  $j \in B$  and  $i \in S'$  as  $\tilde{p}_i^\epsilon = 2z$ , where  $z \in \mathbb{Z}$  is the smallest integer such that  $\tilde{p}_i^\epsilon \geq p_i^S$ . That is, we adjust prices up by  $\epsilon$  so that they are on the grid for valuations.

The argument is almost the same as the previous theorem, except we shift prices instead of valuations. The market clearing is again ensured by the construction of  $\Pi(p)$  and the sellers are optimizing since  $\tilde{p}^\epsilon \geq p_i^S \geq c_i$ . It remains to show that the buyers are optimizing.

We therefore claim that each buyer  $j \in B$  maximizes utility by choosing  $y'_j$  given prices  $\tilde{p}^\epsilon$ . For suppose not; then one of the following cases must hold:

- (i)  $x_{ij} = 1$  for some  $i \in S'$  but there is  $i' \in S'$  such that  $h_{i'j} - \tilde{p}_{i'}^\epsilon \geq h_{ij} - p_i^\epsilon + 1$ ,
- (ii)  $x_{ij} = 0$  for all  $i \in S'$  but there is  $i' \in S'$  such that  $h_{i'j} - \tilde{p}_{i'}^\epsilon \geq 1$ .

In case (i),  $h_{i'j} - \tilde{p}_{i'}^\epsilon = h_{ij} - \tilde{p}_i^\epsilon + 1$  is impossible by construction of  $\tilde{p}^\epsilon$ . Then if  $h_{i'j} - \tilde{p}_{i'}^\epsilon \geq h_{ij} - \tilde{p}_i^\epsilon + 2$  buyer  $j$  can deviate to  $p_{i'j}^B = \tilde{p}_{i'}^\epsilon + 1$  and  $p_{i''j}^B = 0$  for all  $i'' \neq i$  for an extra payoff of

$$(h_{i'j} - p_{i'}^S) - U_j^B(p) \geq 1 > 0,$$

since  $U_j^B(p) \leq (h_{ij} - p_i^S)$ . After the deviation, the clearing house should match  $j$  and  $i'$  since every other match makes zero or negative surplus by the previous step.

Similarly, in case (ii), buyer  $j$  can deviate to  $p_{i'j}^B = p_i^S \leq \tilde{p}_{i'}^\epsilon$  for some positive reward from a tie. The clearing house should then match  $j$  and  $i'$  after the deviation with positive probability.

Finally, since no payoffs have decreased, no players are playing their dominated actions. Any buyer who was buying a good in the Nash equilibrium for some price  $p_i^S$  has a non-negative payoff for the same good for the price  $\tilde{p}^\epsilon$  by Assumption 2.

Hence, in each of the two cases,  $p$  cannot be a Nash equilibrium.

□

## REFERENCES

- ACKERMANN, H., P. W. GOLDBERG, V. S. MIRROKNI, H. RÖGLIN, AND B. VÖCKING (2011): “Uncoordinated two-sided matching markets,” *SIAM Journal on Computing*, 40, 92–106.
- APT, K. R. AND S. SIMON (2015): “A classification of weakly acyclic games,” *Theory and Decision*, 78, 501–524.
- BARRON, E., R. GOEBEL, AND R. JENSEN (2010): “Best response dynamics for continuous games,” *Proceedings of the American Mathematical Society*, 138, 1069–1083.
- BENASSY, J.-P. (1986): “On competitive market mechanisms,” *Econometrica*, 54, 95–108.
- CHEN, B., S. FUJISHIGE, AND Z. YANG (2010): “Decentralized market processes to stable job matchings with competitive salaries,” *KIER Discussion Paper*, 749.
- DEMANGE, G., D. GALE, AND M. SOTOMAYOR (1986): “Multi-item auctions,” *Journal of political economy*, 94, 863–872.
- DUBEY, P. (1982): “Price-quantity strategic market games,” *Econometrica*, 50, 111–126.
- FOSTER, D. AND P. YOUNG (1990): “Stochastic evolutionary game dynamics,” *Theoretical population biology*, 38, 219–232.
- HOFBAUER, J. AND S. SORIN (2006): “Best response dynamics for continuous zero-sum games,” *Discrete and Continuous Dynamical Systems Series B*, 6, 215.

- KLAUS, B., F. KLIJN, AND M. WALZL (2010): “Stochastic stability for roommate markets,” *Journal of Economic Theory*, 145, 2218–2240.
- KLAUS, B. AND J. NEWTON (2016): “Stochastic stability in assignment problems,” *Journal of Mathematical Economics*, 62, 62–74.
- MAS-COLELL, A. (1982): “Perfect competition and the core,” *Review of Economic Studies*, 49, 15–30.
- MILCHTAICH, I. (1996): “Congestion games with player-specific payoff functions,” *Games and economic behavior*, 13, 111–124.
- MONDERER, D. AND L. S. SHAPLEY (1996): “Potential games,” *Games and economic behavior*, 14, 124–143.
- NAX, H. H. (2019): “Uncoupled aspiration adaptation dynamics into the core,” *German Economic Review*, 20, 243–256.
- NAX, H. H. AND B. S. PRADELSKI (2015): “Evolutionary dynamics and equitable core selection in assignment games,” *International Journal of Game Theory*, 44, 903–932.
- NAX, H. H., B. S. PRADELSKI, AND H. P. YOUNG (2013): “Decentralized dynamics to optimal and stable states in the assignment game,” in *52nd IEEE Conference on Decision and Control*, IEEE, 2391–2397.
- NEWTON, J. AND R. SAWA (2015): “A one-shot deviation principle for stability in matching problems,” *Journal of Economic Theory*, 157, 1–27.
- SHAPLEY, L. S. AND M. SHUBIK (1971): “The assignment game I: The core,” *International Journal of Game Theory*, 1, 111–130.
- SIMON, L. K. (1984): “Bertrand, the Cournot paradigm and the theory of perfect competition,” *Review of Economic Studies*, 51, 209–230.
- YOUNG, H. P. (1993): “The evolution of conventions,” *Econometrica: Journal of the Econometric Society*, 57–84.