2: Infinitisemals

Math camp 2018

George Mason University

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- Me: Arthur Dolgopolov,
- Don't hesitate to email me questions! Seriously.
  adolgopo@gmu.edu
- Lecture slides:
  https://arthurdolgopolov.net/teaching/mathcamp
- Wolfram Mathematica (free for students):
  https://cos.gmu.edu/mathematica/

- Quick recap, some more rules for measuring sets. Combinatorics.

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- Axiomatic Real numbers.

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- Axiomatic Real numbers.
- Calculus review: limits, continuity, differentials, integration.

Heavy option:

Vladimir Zorich Mathematical Analysis (Part I)

Light option: 1st chapter in

Mathematical methods for economic theory Martin J. Osborne
https://mjo.osborne.economics.utoronto.ca/index.php/tutorial/index/1/toc

Previously...

Everything is built on set theory.
Graphs of functions, relations, orders are sets.
Sets can be finite countable, infinite countable, infinite uncountable.

We know how to build propositional algebra and how to work with propositions.
Therefore, we also know how to prove things - directly, by contrapositive, by contradiction, by induction, combinatorially, by cases or in any other valid ways.
We know how to measure sets using functions.

We haven't built numbers as a valid object yet.

Review: Measuring more sets

As a recap, let's measure a few more sets.
Possible combinations of 12 donuts of 5 flavors.

Definition

a permutation is the act of arranging the members of a set into a sequence or order

$P(n, k) = n(n-1)...(n-k+1) = \frac{n!}{n-k!}$

Definition

a combination is a selection of items from a collection, such that (unlike permutations) the order of selection does not matter

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$ $k \leq n$ and 0 otherwise.

Proof

Choose k elements in some order. This is a surj map - every element is counted $k!$ times.

So $n \times (n-1)...(n-k+1)/k!$.

In how many ways can I select 5 books from my collection of 100 to bring on vacation?

$$\binom{100}{5}$$

Subset rule:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

BOOKKEEPER rule:

how many sequences can be formed by permuting the letters in the 10-letter word BOOKKEEPER?

$$\binom{n}{k} = \frac{10!}{1!2!2!3!1!1!}$$

For completeness:
Combinations with repetition:

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}$$

If you forgot the intuition behind this, it is just stars and bars:

$$\star\star\star|\star|$$

$$\star|\star\star\star|$$

https://en.wikipedia.org/wiki/Stars_and_bars_(combinatorics)

Part 1: Construction of $\mathbb{R}$

Real numbers $\mathbb{R}$

As a complete ordered field (Synthetic approach)

Axiom system:

$+ : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies:

1_+: a neutral or identity element 0 s.t. $x + 0 = 0 + x = x$ for every $x \in \mathbb{R}$.

2_+: $\forall x \in \mathbb{R}$ there is $-x \in \mathbb{R}$, s.t. $x + (-x) = (-x) + x = 0$

3_+: Associativity $x + (y + z) = (x + y) + z \; \forall x, y, z \in \mathbb{R}$.

Technically, this makes the set with the operation an Abelian group.

Without going into details, a group is a fundamental structure to study symmetry.

Set of rotated squares, and an operation of rotating this square by 90°. This group would satisfy similar axioms.

The hours on a 12-hour clock form a group under addition modulo 12: $3 + 11 = 2$, i.e. $14 = 2pm$.

$\cdot : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.
1.: a neutral or identity element $1 \in \mathbb{R} \setminus 0$ s.t. $x \cdot 1 = 1 \cdot x = x$ for every $x \in \mathbb{R}$.
2.: $\forall x \in \mathbb{R} \setminus 0$ there is $x^{-1} \in \mathbb{R}$, s.t. $x \cdot x^{-1} = x^{-1} \cdot x = 1$
3.: Associativity $x \cdot (y \cdot z) = (x \cdot y) \cdot z \; \forall x, y, z \in \mathbb{R}$.
4.: Commutativity $x \cdot y = y \cdot x \; \forall x, y \in \mathbb{R}$.

Also a group. So far no connection between the two groups.
D: Distributivity
$(x + y) \cdot z = x \cdot z + y \cdot z$

Technically, axioms together make our set a field. A field of real numbers.

Order axioms:

A relation $\leq$ with the following properties:
$0_{\leq} \forall x \in \mathbb{R}(x \leq x)$
$1_{\leq} (x \leq y) \wedge (y \leq x) \implies (x = y)$
$2_{\leq} (x \leq y) \wedge (y \leq z) \implies (x \leq z)$
$3_{\leq} \forall x \in \mathbb{R} \forall y \in \mathbb{R} (x \leq y) \vee (y \leq x)$

Connection of operations to order:
$(x \leq y) \implies (x + z \leq y + z) \forall x, y, z \in \mathbb{R}$.
$(0 \leq x) \wedge (0 \leq y) \implies (0 \leq x \cdot y)$. Completeness:

If X and Y are nonempty subsets of $\mathbb{R}$ having the property that $x \leq y\ \forall x \in X, y \in Y$ then there is $c \in \mathbb{R}$, s.t. $x \leq c \leq y$ $\forall x \in X, y \in Y$.

Consequences:

1

There is only one zero in the set of real numbers

2

Each element has a unique negative

3

$a + x = b$ has a unique solution $x = b + (-a)$

Let's prove some of them.

Consequences:

1

There is only one multiplicative unit in the set of real numbers

2

Each element has a unique inverse (reciprocal)

3

$a \cdot x = b$ has a unique solution $x = b(a)^{-1}$

Consequences:

1

$0 \cdot x = x \cdot 0 = 0$

2

$(x \cdot y = 0) \implies (x = 0) \vee (y = 0)$

3

$-x = (-1)x$

Consequences:

1

$\forall x, y \in \mathbb{R}$
Exactly one of the three holds:
$x < y, x = y, x > y$

2

$\forall x, y, z \in \mathbb{R}$

$$(x < y) \wedge (y \leq z) \rightarrow (x < z)$$

$$(x \leq y) \wedge (y < z) \rightarrow (x < z)$$

3

$\forall x, y, z \in \mathbb{R}$

$$(0 < x) \wedge (0 < y) \implies (0 < xy)$$

4

$$(0 < 1)$$

Definition

A set $X \subset \mathbb{R}$ is bounded above if $\exists c \in \mathbb{R}$,s.t. $x \leq c$ for all $x \in X$.

c is called the upper bound.

Definition

A set bounded above and below is bounded.

$a = \max X := (a \in X \wedge \forall x \in X(x \leq a))$
$a = \min X := (a \in X \wedge \forall x \in X(a \leq x))$
$X = \{x \in \mathbb{R} | 0 \leq x < 1\}$ has a min but not a max.

Least upper bound or Exact upper bound

$(s = sup\ X) := \forall x \in X, (x \leq s) \wedge (\forall s' < s\ \exists x' \in X, \text{ s.t. } s' < x')$

Greatest lower bound or Exact lower bound

$(i = inf\ X) := \forall x \in X, (i \leq x) \wedge (\forall i' > i\ \exists x' \in X, \text{ s.t. } x' < i')$

Lemma

(X bounded above) $\implies \exists!\ \inf X$

Proof:

Let $X \subset \mathbb{R}$, $Y = \{y \in \mathbb{R} | \forall x \in X (x \leq y)\}$.
$X \neq \emptyset$, $Y \neq \emptyset$. By completeness $c \in \mathbb{R}$, s.t.
$\forall x \in X\ \forall y \in Y, x \leq c \leq y$. Then $c = \min Y = \sup X$.

But synthetic approach without set theory is cheating.

Let's start by constructing $\mathbb{N} = 0, 1, 2 \ldots$.

Our axioms (specifically, axiom of the empty set) allow me to define a zero

$0 = \emptyset$

A few extra ZFc axioms will allow me to iteratively define:

$V_0 = \emptyset$

$V_1 = \{\emptyset\}$

$V_2 = \{\emptyset, \{\emptyset\}\}$

$V_3 = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

...

We will also need Foundation from ZFc: $x \notin x$ for any set x.

Definition (ordinals)

An ordinal number is a set that is transitive and is well-ordered by the relation $a < b \iff a \in b$.

Infinite ordinals will require some more work, but one can use Replacement and Infinity to define the first cardinal $\omega = \{0, 1, 2...\}$. More formally, this is a set of all finite ordinals and its cardinality is $\aleph_0$.

In fact definitions of cardinal and ordinal numbers start to diverge for infinite sets. This is behind the term "cardinality" for the size of sets. Cardinal numbers are used for counting (size), ordinal numbers are used for ordering (sets in our definitions are well-ordered).

We will stop here. I will only hint at how to formally construct reals from here. First build integers $\mathbb{Z}$.

Take $\mathbb{N} \times \mathbb{N}$ and let every (a, b) represent an integer $a - b$.

Now build $\mathbb{Q}$. Take $\mathbb{Z} \times (\mathbb{Z} \setminus 0)$ and let every (a, b) represent a/b.

The construction of reals from rationals takes a bit longer. The canonical way uses so-called Dedekind cuts, which are not difficult, but are very far from economics program to build up here.

We will instead do the conventional Cauchy sequences approach.

"In measuring real physical quantities we obtain sequences of approximate values with which one must then work"

For example $\pi = 3.14159...$
or rather $|\pi - 3.14159| < 10^{-5}$.

$f : \mathbb{N} \to \mathbb{R}$ is called a sequence.

An open interval containing the point $x \in \mathbb{R}$ will be called a neighborhood of this point.

$A \in \mathbb{R}$ is a limit of numerical sequence $\{x_n\}$ if for every neighborhood V(A) of A there is N such that all terms having index larger than N belong to the neighborhood $V(A)$.

$A \in \mathbb{R}$ is a limit of numerical sequence $\{x_n\}$ if for any $\epsilon > 0$, there is an index N such that $|x_n - A| < \epsilon$ for any $n > N$.

$$\lim_{n \to \infty} x_n = A$$

A sequence that has a limit is called convergent, otherwise it is divergent.

Don't confuse this with series.
Any infinite sequence defines an infinite series or simply series, which is just the sum of all terms in the sequence.

$$\sum_{i=1}^{\infty} a_i$$

Long considered a paradox, it can be calculated easily now with a limit:

$$\sum_{i=1}^{\infty} a_i = lim_{n \to \infty} \sum_{i=1}^{n} a_i$$

Examples of series

π:

$$\sum_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$$

Geometric series:

$$\sum_{n=1}^{\infty} z^n$$

(converges iff $|z| \leq 1$)

Alternating series:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = ln(2)$$

A fun property:

if $\sum |a_n|$ converges then any reordering of a_n also converges to the **same** limit.

A secret weapon when working with series: http://oeis.org

(wiki is not bad on this topic)
https://en.wikipedia.org/wiki/Indeterminate_form

Indeterminate forms:
$\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$, 1^{∞}, $\infty - \infty$, 0^0, ∞^0.

$$\lim_{x \to c} \frac{f(x)}{g(x)}$$

Indeterminate form CAN (often) be evaluated. It does not mean the value does not exist.
As usual, algebra and substitution are the most powerful tools.

A general approach:

L'Hôpital's rule

f and g differentiable on an open interval I except possibly at a point $c \in I$, $g'(x) \neq 0 \forall x \in I, x \neq c$,

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

if the former is an indeterminate form and the latter exists.

Economics application (Consumer theory) of limits:
CES function

$$Y = A(\sum \alpha_j Y_j^{\gamma})^{\frac{1}{\gamma}}$$

$$ln(Y) = ln(A) + \frac{F(\gamma)}{G(\gamma)}$$

Derive limits at $\gamma \to 0, 1, -\infty$

We can now introduce irrational numbers
A sequence of rational approximations to π is not converging to any rational number, but is converging to something.
This is a Cauchy sequence.

Definition (Cauchy sequence)

A rational sequence (a_n) is a Cauchy sequence if the difference between its terms tends to 0. Given any small rational number q, there is a natural number n such that $\forall m', n' > n, |a_n - a_m| < q$.

We can now use this to build $\mathbb{R}$. The problem is that there are multiple sequences that converge to the same number, so we will end up with duplicates in our sets. Loosely speaking, grouping 'duplicates' into sets is a very common operation.

Definition (Equivalence relations)

Let S be a set of (mathematical) objects. A relation R among pairs of elements of S is an equivalence relation if:

1. (Reflexivity): for any $s \in S$ sRs.
2. (Symmetry): for any $s, t \in S$, if sRt then tRs.
3. (Transitivity) for any $s, t, r \in S$, if sRt and tRr then sRr.

"Indifference" is an equivalence relation. Each set of objects with the same utility, i.e. among which the person is indifferent, forms an indifference class.

"What remains", the partitioning itself is a quotient $S/\sim$.

360° rotations from before form equivalence classes. Modular arithmetic.

More examples: $\sim$ is "the same color", equivalence classes consist of objects of the same color, $X/\sim$ the set of all colors. Equivalence classes are disjoint. (Easy to proof).

We now can finally define $\mathbb{R}$
We will use this equivalence relation:
We will say (a_n) (b_n) if

$$a_n - b_n \to 0$$

This is an equivalence relation (Another easy proof).
$\mathbb{R}$ are the equivalence classes of this relation. The fact that operations are well-defined follows (see details at e.g. http://www.math.ucsd.edu/~tkemp/140A/Construction.of.R.pdf).

A metric space is a set for which distances between all members of the set are defined.

A metric space M is called complete or Cauchy if every Cauchy sequence of points in M has a limit that is also in M.
there are no "points missing" from it (inside or at the boundary).

Examples: $\mathbb{Q}$ is not complete, $x_1 = 1$, $x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$. It uses rational numbers, but converges to an irrational number $\sqrt{2}$.
$(0, 1)$ is not complete, $x_n = \frac{1}{n}$ is Cauchy, but the limit would be 0, outside of this space.

A normed vector space is a metric space with the metric $d(x, x') = ||x - x'||$

A complete normed vector space is called a Banach space. E.g. $\mathbb{R}^n$.

Part 2: Infinitesimals

We are familiar with functions (functions $\supset$ relations $\supset$ sets). We will now focus on continuous functions.

a **continuous** function is a function for which sufficiently small changes in the input result in arbitrarily small changes in the output.

$$lim_{x \to c} f(x) = f(c)$$

a **differentiable** function of one real variable is a function whose derivative exists at each point in its domain.

$$f'(a) = lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

partial derivative of a function of several variables is its derivative with respect to one of those variables, with the others held constant

The **boundary** of a subset S is the set of points which can be approached both from S and from the outside of S.
A set is **open** if it doesn't contain any of its boundary points.
It is **closed** otherwise.

Theorem (Intermediate Value)

If a continuous function, f, with an interval, $[a, b]$, as its domain, takes values $f(a)$ and $f(b)$ at each end of the interval, then it also takes any value between $f(a)$ and $f(b)$ at some point within the interval.

Theorem (Maximum Value (Weierstrass))

If a continuous function, f, is continuous on a closed interval, it is bounded on this interval. Moreover, there is a point in the interval, where the maximum and minimum values are achieved.

Trick question:
A function with discrete domain, e.g. natural numbers. Can it be continuous?

A metric space is a set for which distances between all members of the set are defined.
Slightly more rigorous definition of continuity (for metric spaces):

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in E \\ ((d_x(x, p) < \delta) \implies (d_y(f(x), f(y)) < \epsilon))$$

Any $0 < \delta \leq 1$ would work for $\mathbb{N}$.

Discrete topology

Every function $\mathbb{N} \to Y$ is continuous. Every function that maps from discrete domain D is continuous.

Proof:
For all $\epsilon > 0$, $x \in D$, there is δ, s.t. $y \in D$ is close to X, i.e. $|x - y| < \delta$ then $f(y)$ is close to $f(x)$, i.e. $|f(x) - f(y)| < \epsilon$. But there is some δ s.t. the only point within δ from x is x.

A function is continuous means that limits of all convergent sequences are preserved. A convergent sequence in a discrete set is a constant, and such sequences are always preserved by any function.

Claim (1)

Any continuous function is differentiable.

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Claim (4)

If a function is not continuous, it is not differentiable.

Claim (1)

Any continuous function is differentiable.

Claim (2)

A differentiable function has to be continuous.

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If a function is not differentiable, it is not continuous.

Claim (4)

If a function is not continuous, it is not differentiable.

Claim (5)

Continuous function is differentiable except for a set of isolated points.

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Claim (5)

Continuous function is differentiable except for a set of isolated points.

The most interesting counterexample to claim 5:
https://en.wikipedia.org/wiki/Weierstrass_function

Definition (Metric Spaces)

A metric space is a set M for which distances $d : M \times M \to \mathbb{R}$ between all members of the set are defined.

Our 3D Euclidean space? Wealth of top bankers?
Colors? Students?

$$d(red, blue)?$$

Definition (Metric Spaces)

A metric space is a set M for which distances $d : M \times M \to \mathbb{R}$ between all members of the set are defined.

Our 3D Euclidean space? Wealth of top bankers?
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$$d(red, blue)?$$

$$= 3?$$

green?

Discrete topology

Every function $\mathbb{N} \to Y$ is continuous.

a metric space is a set for which distances between all members of the set are defined.
Slightly more rigorous definition of continuity (for metric spaces):

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall x \in E$$
$$((d_x(x, p) < \delta) \implies (d_y(f(x), f(y)) < \epsilon))$$

Let $0 < \delta \leq 1$.
Outside of our domain, notions of continuity get weird fast.
General property for topologies where every subset is open, that is singletons without accumulation points. This allows us to get *max* and *min*, although intermediate value theorem still fails (one needs **connectedness**).

Differentiation

Derivative:

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

If the function is not continuous, the limit does not exist.

To review common derivative rules, take any derivative tables, e.g. http://tutorial.math.lamar.edu/pdf/Common_Derivatives_Integrals.pdf
Similarly for limit properties, there are numerous tables to look them up, e.g. http://tutorial.math.lamar.edu/Classes/CalcI/LimitsProperties.aspx

Lagrange's Notation is to write the derivative of the function $f(x)$ as

$$f'(x)$$

Leibniz's Notation is to write the derivative of the function f as

$$\frac{df}{dx}$$

Newton's Notation is to write the derivative of y using a dot

$$\dot{y}$$

Euler's Notation is to use a capital D i.e.

$$D_x f(x)$$

Some rules:

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$\frac{d}{dx}[f(g(x))] = \frac{df}{dg} \times \frac{dg}{dx}$$
$$\frac{d}{dx}[e^x] = e^x$$
$$\frac{d}{dx}[lnx] = 1/x$$

$$\frac{d}{dx}[e^x] = e^x$$

Proof:

$$D_x[ln(e^x)] = D_x(x)$$

$$\frac{1}{e^x} D_x(e^x) = 1$$

$$D_x(e^x) = e^x$$

If starting from ground up:

http://us.metamath.org/mpegif/dvexp.html

$f(x) = sin(x)$
$f'(x) = cos(x)$

$$\lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h} = \lim_{h \to 0} \frac{2\sin(h/2)\cos(x+h/2)}{h} =$$

$$= \lim_{h \to 0} cos(x + \frac{h}{2}) \lim_{h \to 0} \frac{\sin(h/2)}{h/2} = \cos x$$

Last by the fact that $\sin t \sim t$ at $t \to 0$.

A very useful application is optimization, but we will devote a separate day to that.
A (Riemann) integral. For more general notions see https://en.wikipedia.org/wiki/Lebesgue_integration.
Instead of a formal definition, we will stick with intuition and a theorem.

An example: surface area and circumference of a disk.

An example: surface area and circumference of a disk.
Sphere?

An example: surface area and circumference of a disk.
Sphere? N-sphere?

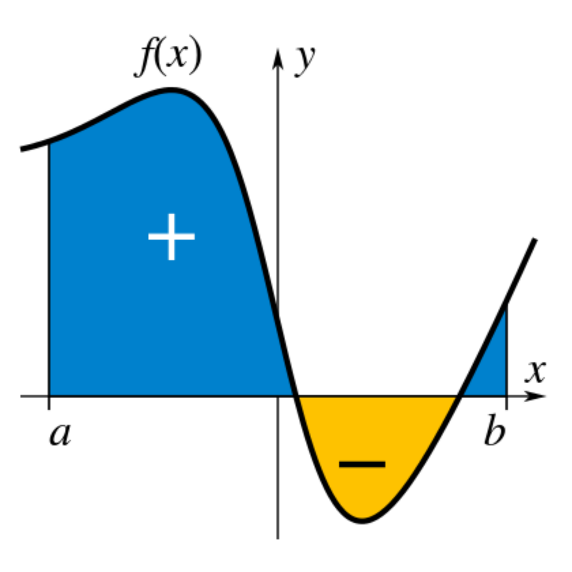

Figure: Integration

Fundamental theorem of calculus, Newton-Leibniz

1. $F'(x) = f(x)$

2. $\int_a^b f(x)dx = F(b) - F(a)$

where f is some real-valued function on closed interval $[a, b]$, and F - antiderivative of f in $[a, b]$.

(it is a little underwhelming really)

Fundamental theorem of calculus, Newton-Leibniz

1. $F'(x) = f(x)$

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where f is some real-valued function on closed interval $[a, b]$, and F - antiderivative of f in $[a, b]$.

Two tricks:
Integration by parts

$$\int u \frac{dv}{dx} dx = uv - \int v \frac{du}{dx} dx$$

Variable substitution

$$\int_{\phi(\alpha)}^{\phi(\beta)} f(x) d(x) = \int_{\alpha}^{\beta} f(\phi(t))\phi'(t) d(t)$$

Example.

$$\int_{-1}^{1} \sqrt{1 - x^2} dx = \frac{\pi}{2}$$

Substitute $x = \sin t$.

See Zorich, p. 367

Practice - "make a mathematician twitch".

Why denoting sequence $\{x_n\}$ a little disturbing for set theorists?
$f : \mathbb{N} \to \mathbb{R}$ is the best, but what is $\mathbb{N}$?
What does $a \wedge b \to p$ stand for (remember the order of precedence)?
f^{-1}. This is never confusing in practice.
What is 0^0?

Math is a language, and readers can interpret things. Use your best judgment when abusing notation (and try not to).
Thank you!