# 5: Additional Methods

Math camp 2019

George Mason University

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- Me: Arthur Dolgopolov,
- Don't hesitate to email me questions! Seriously.
  adolgopo@gmu.edu
- Lecture slides:
  https://arthurdolgopolov.net/teaching/mathcamp
- Wolfram Mathematica (free for students):
  https://cos.gmu.edu/mathematica/

- A good quick resource is

Mathematical methods for economic theory by Martin J. Osborne, Chapter 8 (online and free) https://mjo.osborne.economics.utoronto.ca/index.php/tutorial/index/1/dee/c

- To actually get into solving complicated differential equations by hand - pick up a book, e.g.

Zhang, Wei-Bin. Differential equations, bifurcations, and chaos in economics. Vol. 68. World Scientific Publishing Company, 2005.

# Part 1: Differential equations

A **differential equation** is an equation that includes an unknown function and its derivatives.

Warm-up:

$$\dot{x}(t) = 2$$

$$\dot{x}(t) = 2t$$

$$\dot{x}(t) = x(t)$$

$$\dot{x}(t) = kx(t)$$

Economists usually express a rate of change of the current state as a function of the current state. This is a model of exponential growth.
e.g. GDP changes proportional to current GDP (think Crusonia plant)

$$\dot{x}(t) = gx(t)$$

$\dot{x}$ - derivative of $x(t)$ with respect to $t$, rate of change.
$\dot{x}/x$ - growth rate.
**Solution** is a function: $x(t) = x(0)e^{gt}$. That is $x$ changes exponentially.

**Theorem**

*Solution to* $\dot{x}(t) = kx(t)$ *is* $x(t) = x_0 e^{kt}$ *with* $x_0 = x(0)$.

There are **general** solutions with parametric constants and **partial** solutions with specific initial point.

A very convenient way to look at differential equations are direction fields. It is like plotting function from really tiny pieces.

An example: Newton's Second Law

$$m\frac{dv}{dt} = F(t, v)$$

two forces:
$F_G = mg$ and $F_A = -\gamma v$ Simplify

$$\frac{dv}{dt} = g - \frac{\gamma v}{m}$$

$$\frac{dv}{dt} = 9.8 - 0.196v$$

Sketch of proof.
Write in different notation

$$\frac{dy}{dt} = ky$$

$$\frac{dy}{y} = kdt$$

$$\int \frac{dy}{y} = \int kdt$$

$$ln\ y = \lambda t + c$$

$$y = e^c e^{\lambda t}$$

Let $C = e^c$. Since $y(0) = C$, we use $y_0 = y(0) = C$

$$y = y_0 e^{\lambda t}$$

This is a general solution. If the problem had an extra equation specifying the boundary condition, i.e. $y(0) = 3$, we would get a "particular solution".

Continuous compounding formula (or force of interest):

$$P(t) = P_0 e^{rt}$$

Why? Where did exponent come from in the growth equation?
Usual compounding:

$$P' = P(1 + \frac{r}{n})^{nt}$$

$$e = lim_{n\to\infty}(1 + \frac{1}{m})^m \approx 2.71828$$

$$\lim_{n\to\infty} P\left(1 + \frac{r}{n}\right)^{nt} = \lim_{m\to\infty} P\left(1 + \frac{1}{m}\right)^{mrt}$$

$$= P\left(\lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^{m}\right)^{rt} = Pe^{rt}$$

Perhaps, "splitting" $\frac{dy}{dt}$ makes you uncomfortable. Two solutions:

- 1. Treat $dy$ and $dt$ as infinitesimals. It is possible to build rigorous foundation for mnemonic of "splitting" the Leibniz notation from that.

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- 1. Treat $dy$ and $dt$ as infinitesimals. It is possible to build rigorous foundation for mnemonic of "splitting" the Leibniz notation from that.
- 2. Don't split. Use the chain rule (or, equivalently, integration by parts).

Exponential growth is everywhere. compounding, GDP/population growth, radioactive decay/h$\textcircled{\lambda}$lf-life, heat transfer, networks, Moore's law of processor power, predator-prey dynamics etc.

**Problem**

Suppose bacteria grow exponentially with $\dot{x} = kx$. At the end of 3 hours there are 10k bacteria. At the end of 5 hours there are 40k. How many were present initially?

- $y' = ky$ so $y = y_0 e^{kt}$

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- Now just plug in $y = y_0 e^{tln2}$ with $t = 3$.

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- Now just plug in $y = y_0 e^{tln2}$ with $t = 3$.
- $$10000 = y_0 e^{3ln2} = y_0 8$$
- $$y_0 = 1250.$$

We would like to solve more complicated DE.
More generally,

**Definition**

Ordinary differential equation - solutions are functions of one independent variable, i.e. $t$

$$G(t, x(t), x'(t), x"(t), ..., x(n)(t)) = 0$$

If we would have multivariate functions we would be dealing with **partial DE** (not for math camp).

**Definition**

Autonomous differential equation - variable $t$ does not explicitly appear.

$$\dot{x}(t) = ax(t) + b$$

**Definition**

$n$th order differential equation - derivatives up to $n$th.

**Definition**

$n$th power differential equation - highest power of a derivative in DE is $n$.

## Theorem (Existence for first-order ordinary differential equation)

*If $F$ is a function of two variables that is continuous at $(t_0, x_0)$ then*
*$\exists a > 0$ and a continuously differentiable function $x$ of a single variable defined on the interval $(t_0 - a, t_0 + a)$ that solves*

$$x'(t) = F(t, x(t))$$

$$x(t_0) = x_0$$

*for all $t \in (t_0 - a, t_0 + a)$.*
*If in addition $\frac{\partial F}{\partial x}$ is continuous on an open rectangle containing $(t_0, x_0)$ then solution is unique.*

# Non-unique solution

$$x'(t) = (x(t))^{1/2}$$
$$x(0) = 0$$

$$x(t) = 0$$

$$x(t) = (t/2)^2$$

### Definition

If for some initial condition a first-order initial value problem has a solution that is a **constant** function (independent of t), the value of the constant is an **equilibrium** or **stationary state** of the associated differential equation.

Consider $x'(t) + x(t) = 2$ with initial value $x(0) = x_0$.
Solution is $x(t) = (x_0 - 2)e^{-t} + 2$.
Verify that it is a solution.
For other values of $x_0$ we get convergence to 2, we call such equilibria **stable**.

## Definition

A **linear** first-order ordinary differential equation is a first-order ordinary differential equation that may be written in the form

$$x'(t) + a(t)x(t) = b(t)$$

for functions $a$ and $b$ of a single variable.

The general solution of the linear first-order ordinary differential equation

$$x'(t) + ax(t) = b(t),$$

where $a$ is a constant and $b$ is a continuous function, is given by

$$x(t) = e^{-at}[C + \int^t e^{as} b(s) ds]$$

If $b$ is a constant function, with $b(s) = b$ for all $s$, and $a \neq 0$, then this general solution is

$$x(t) = Ce^{-at} + \frac{b}{a}$$

$\int^t$ is the antiderivative evaluated at $t$

Example

$$x'(t) + 2x(t) = 6.$$

Use previous result.
Is the solution stable?

The linear first-order ordinary differential equation

$$x'(t) + ax(t) = b$$

with $a \neq 0$ has a unique equilibrium, $\frac{b}{a}$. This equilibrium is globally stable if $a > 0$ and is unstable if $a < 0$.

Second-order DE:
Arguably, the most important trick in optimization and DE: **variable substitution**.

Cycles of derivatives.

$$f(x) = f''(x)$$

$f(x) = e^x$, but this is too easy
An answer for cycles of 4 is also easy. ?

$$\frac{d}{dx}a' = 0$$

$$\frac{d}{dx}x = 1$$

$$\frac{d}{dx}au = a\frac{du}{dx}$$

$$\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$$

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$\frac{d}{dx}f(u) = \frac{d}{du}f(u)\frac{du}{dx}$$

$$\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$$

$$\frac{d}{dx}ln(u) = \frac{du}{dx}\frac{1}{u}$$

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}$$

$$\frac{d}{dx}a^u = a^u lna\frac{du}{dx}$$

$$\frac{d}{dx}sinu = cosu\frac{du}{dx}$$

$$\frac{d}{dx}cosu = -sinu\frac{du}{dx}$$

$$f(x) = cos(x)$$
$$f'(x) = -sin(x)$$
$$f''(x) = -cos(x)$$
$$f'''(x) = sin(x)$$
$$f''''(x) = f(x)$$

.

$$f(x) = f''(x) \wedge f(x) \neq f'(x)$$

The solution is

$$f(x) = c_1 e^x + c_2 e^{-x}$$

The general solution is a bit trickier and is in $\mathbb{C}$:
https://math.stackexchange.com/questions/1598116/derivative-cycles-of-length-8
https://math.stackexchange.com/questions/1256140/for-what-functions-is-y-y

find all $f$ such that $y(x) = y''(x)$
Solution by variable change kung fu:

$$y'' + y' = y + y'$$
$$(y' + y)' = y' + y$$

Similarly

$$(y' - y)' = y'' - y' = y - y' = -(y' - y)$$

$$y' + y = Ce^{x}$$
$$y' - y = De^{-x}$$

Sum the equations

$$y = Ae^{x} + Be^{-x}$$

The following example is from Steven H. Strogatz, Nonlinear Dynamics and Chaos, Addison Wesley, 1994 via William Cherry.

- $R(t)$ - Romeo's affection for Juliet at time $t$. $J(t)$ - Juliet's affection for Romeo at time $t$.
- We will say $R > 0$ corresponds to positive affection i.e., love, for Juliet, and $R < 0$ corresponds to negative affection, i.e., hate.
- Then, $dR/dt$ and $dJ/dt$ represent how Romeo and Juliet's affections for each other are changing at an instant in time.
- Suppose the change in Romeo's affection for Juliet is proportional to how much affection he already has for her, and similarly for Juliet.

In equations, this could be written:

$$\frac{dR}{dt} = \lambda_1 R$$

$$\frac{dJ}{dt} = \lambda_2 J$$

It is a system, but a system of completely independent equations. Solve each one:

$$R = R_0 e^{\lambda_1 t}$$

$$J = J_0 e^{\lambda_2 t}$$

But, these are **linear** differential equations. In matrix form:

$$\begin{bmatrix} \frac{dR}{dt} \\ \frac{dJ}{dt} \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \times \begin{bmatrix} R \\ J \end{bmatrix}$$

What is the intuition for $\lambda$s without solving?

Can we plot a phase diagram? The arrows show how Romeo and Juliet's affections will change.

**R-axis is an "attractor."**

Of course, we want a more difficult puzzle:

$$\begin{bmatrix} \frac{dR}{dt} \\ \frac{dJ}{dt} \end{bmatrix} = \begin{bmatrix} 0 & a \\ b & 0 \end{bmatrix} \times \begin{bmatrix} R \\ J \end{bmatrix}$$

Think of different combinations of $a$ and $b$.
We will show the repeller and attractor are directly related to the eigenvector of the matrix. (They are scalar multiples of eigenvectors - **eigendirections**)

1. Solve characteristic equation
2. Find eigenvalues
3. Solve for eigenvectors (2 eigenvalues, one eigenvector for one eigenvalue)

Two equally cautious lovers.

$$\begin{bmatrix} \frac{dR}{dt} \\ \frac{dJ}{dt} \end{bmatrix} = \begin{bmatrix} -a & b \\ b & -a \end{bmatrix} \times \begin{bmatrix} R \\ J \end{bmatrix}$$

$a > 0, b > 0.$

Romeo and Juliet are both affraid of their own feelings toward the other, so the more Romeo loves Julliet, the more he pulls back, and similarly for Juliet. On the other hand, they respond positively to the other's affection for them, so the more Juliet likes Romeo, the more he tends to like her.

$$\frac{dR}{dt} = aJ$$
$$\frac{dJ}{dt} = -bR$$

Imaginary eigenvalues

$$\begin{bmatrix} \frac{dR}{dt} \\ \frac{dJ}{dt} \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & -1 \end{bmatrix} \times \begin{bmatrix} R \\ J \end{bmatrix}$$

Complex eigenvalues

What are complex numbers? (see next slide)

- (1545)

- (1545)
- Have form $a + bi$.

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- $a$ is real part.
- $bi$ is imaginary part.
- $i$ is the solution to $x^2 = -1$.

Eigenvalues are important:

- Linear algebra: They help understand a linear transformation.
- Differential equations: They point to attractors/repellers and help solve differential equations.
- Optimization: They are directly connected to convexity/concavity of multivariate functions.

Part 2: Probability

## Definition

A countable **sample space** $S$ is a nonempty countable set.
An element $\omega \in S$ is called an **outcome**. A subset of $S$ is called an **event**.
A **probability function** on a sample space $S$ is a total function $Pr : S \to R$ such that

- $Pr[\omega] \geq 0$ for all $\omega \in S$.
- $\sum_{\omega \in S} Pr[\omega] = 1$

A sample space together with a probability function is called a **probability space**. For any event $E \in S$, the probability of $E$ is

defined to be the sum $\sum_{\omega \in E} Pr[\omega]$.

Most of the rules and identities that we have developed for finite sets extend very naturally to probability.

An immediate consequence of the definition of event probability is that for disjoint events E and F: $Pr[E \cup F] = Pr[E] + Pr[F]$.
It follows that $Pr[A] + Pr[\neg A] = 1$. (Why?)

$Pr[B - A] = Pr[B] - Pr[A \cap B]$, (Difference Rule)
$Pr[A \cup B] = Pr[A] + Pr[B] - Pr[A \cap B]$, (Inclusion-Exclusion)
$Pr[A \cup B] \leq Pr[A] + Pr[B]$, (Boole's Inequality)
If $A \subseteq B$, then $Pr[A] \leq Pr[B]$. (Monotonicity Rule)

Let's stop on Boole's Inequality:

## Definition

Mutual independence

$$Pr[A_1 \cap A_2] = Pr[A_1]Pr[A_2]$$

$$Pr[A_3 \cap A_2] = Pr[A_3]Pr[A_2]$$

$$Pr[A_1 \cap A_2 \cap A_3] = Pr[A_1]Pr[A_2]Pr[A_3]$$

etc..

Example: 3 flips.

$A_1$: coin 1 matches coin 2 $A_2$: coin 2 matches coin 3 $A_3$: coin 3 matches coin 1.

Any two of $A_i$ events are mutually independent, but not all three.

$$Pr[A_1 \cap A_2 \cap A_3] = Pr[HHH] + PR[TTT] = \frac{1}{4} \neq \frac{1}{8} = \prod_i Pr[A_i]$$

## Definition

A finite probability space S is said to be **uniform** if $Pr[\omega]$ is the same for every outcome $\omega \in S$.

$$Pr[E] = \frac{|E|}{|S|}$$

Suppose that you select five cards at random from a standard deck of 52 cards. What is the probability of having a full house?

$$2\clubsuit, 2\diamondsuit, 2\spadesuit, J\heartsuit, J\diamondsuit$$

1. The rank of the triple, which can be chosen in 13 ways.
2. The suits of the triple, which can be selected in $\binom{4}{3}$ ways
3. The rank of the pair, which can be chosen in 12 ways.
4. The suits of the pair, which can be selected in $\binom{4}{2}$ ways

1. The rank of the triple, which can be chosen in 13 ways.
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$$|S| = \binom{52}{5}$$

$$|E| = 13\binom{4}{3}12\binom{4}{2}$$

So $P = \frac{18}{12495}$

Hands with two pairs?

$$3\clubsuit, 3\diamondsuit, Q\spadesuit, Q\heartsuit, J\spadesuit$$

Infinite probability spaces are fairly common. For example, two players take turns flipping a fair coin. Whoever flips heads first is declared the winner. What is the probability that the first player wins?

$$P = \frac{1}{2} + \frac{1}{8} + \frac{1}{32} ... = \frac{1}{2} \sum (\frac{1}{4})^n = 0.5 \frac{1}{1 - 0.25} = \frac{2}{3}$$

Second player? $\frac{1}{3}$.
Sample space is the infinite set $S = \{T^n H | n \in \mathbb{N}\}$
Probability function: $Pr(T^n H) = \frac{1}{2^{n+1}}$
All the probabilities are nonnegative and that they sum to 1.

Finite (first $n$ elements)

$$a + ar + ar^2 ... + ar^{n-1} = a\frac{1 - r^n}{1 - r}$$

Infinite

$$a + ar + ar^2 ... = \frac{a}{1 - r}$$

P.S. you can plot systems of DE in Wolfram Alpha online:
http://www.wolframalpha.com/input/?i=streamplot%7B%7B2x,-y%7D,%7Bx,-1,1%7D,%7By,-1,1%7D%7D

https://www.wolframalpha.com/input/?i=y%22(z)+%2B+sin(y(z))+%3D+0&lk=3
streamplot{{2x,-y},{x,-1,1},{y,-1,1}}

*Thank you*