Some problems

August 22, 2019

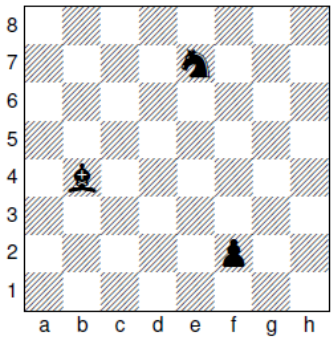
1. Sets

In how many different ways can we place a pawn (P), a knight (N), and a bishop (B) on a chessboard so that no two pieces share a row or a column? A valid configuration is shown in Figure (a), and an invalid configuration is shown in Figure (b).

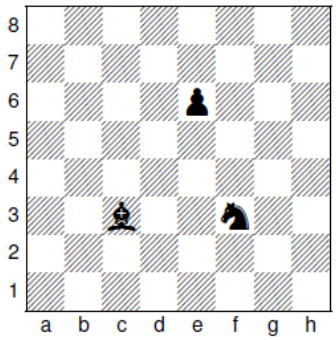
Check your answer on page 620. `https://courses.csail.mit.edu/6.042/spring17/mcs.pdf`

2. Difficult sets

You are about to write down a strategy for your friend to play rock-paper-scissors. It needs to define a move (rock, paper or scissors) for every "history" of play. That is in the first round it has to be the instruction to perform one of the moves. In the second round it has to be the instruction to perform one of the moves for every possible move that the opponent could have made in the first round. In the third round, the instruction should consider all possible moves by the opponent in the first two rounds, etc.. Find the general formula for the number of such strategies given n rounds.



(a) valid



(b) invalid

Option 1: Calculate the number of possible recommendations for each round (not the total) for $n = 1, 2, 3...$ and look for the sequence on the `https://oeis.org/` for the closed form.

Option 2: Work out the closed form solution.

3. Logic

Translate the following sentence into a predicate formula: There is a student who has e-mailed at most n other people in the class, besides possibly himself.

You may use the predicate $\boxtimes(x, y)$ meaning that "x has sent e-mail to y."

4. Matrix arithmetics

Write down any 9 numbers in a 3×3 matrix. E.g.

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$$

Find the determinant, eigenvalues, eigenvectors of the matrix.

Check your answer by typing eigenvalues {{1,-3,3},{3,-5,3},{6,-6,4} into wolfram alpha:

`https://www.wolframalpha.com/input/?i=eigenvalues+%7B%7B1,-3,3%7D,%7B3,-5,3%7D,%7B6,-6,4%7D%7D`

5. Linear algebra

The matrix below represents the roads connecting cities, i.e. if element $a_{ij} = 1$ there is a road going from i to j (roads can be one way). Use matrix multiplication to find a pair of cities (c_1, c_2), such that a traveller would require at least 3 days to reach c_2 from c_1. In how many ways would he be able to do it if he had 4 days (he can not stop - he has to travel every night)?

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

6. Orthogonal projections

In econometrics we are often dealing with estimation errors, which can be written in matrix notation as $e = y - X\hat{\beta}$. Most simple applications try to minimize the squared error. Ignoring the meaning behind the symbols, try to mathematically expand the equation for the squared error $e^T e$ (multiply

the terms and simplify). Now try to find $\hat{\beta}$ that minimizes this expression (by taking a derivative). Check your answers with Stanford https://web.stanford.edu/~mrosenfe/soc_meth_proj3/matrix_OLS_NYU_notes.pdf

7. Integration

Consider the following function:

$$Y = A(\alpha_1 Y_1^{\gamma} + \alpha_2 Y_1^{\gamma})^{1/\gamma}$$

Take the limits of Y at $\gamma \to 1, 0, -\infty$

You can check your answers with this:

http://bergholt.weebly.com/uploads/1/1/8/4/11843961/a_note_on_ces_functions3.pdf

8. Optimization

Solve the following maximization problem:

$$max_{x,y}(\sqrt{x} + y)$$

s.t. $px + y \leq I, x \geq 0, y \geq 0$

where $p > 0, I > 0$

Prove that KKT are necessary and sufficient, i.e. conditions from https://mjo.osborne.economics.utoronto.ca/index.php/tutorial/index/1/osm/t hold.

This is problem 7 at https://mjo.osborne.economics.utoronto.ca/index.php/tutorial/index/1/nnc/x, see the the link for the solution.

9. Differential equations

Find the general solution of the system of equations

$$x'(t) = 4y(t)$$

$$y'(t) = -x(t) + 4y(t)$$

Check your answers at https://mjo.osborne.economics.utoronto.ca/index.php/tutorial/index/1/sim/x or plot the diagram https://www.wolframalpha.com/input/?i=streamplot%7B%7B4y%2C-x%2B4y%7D%2C%7Bx%2C-1%2C1%7D%2C%7By%2C-1%2C1%7D%7D